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PRÜFER ANALYSIS OF PERIODIC SINGULAR STURM–LIOUVILLE PROBLEM WITH PIECEWISE CHARACTERISTIC

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Abstract

Prüfer transformation is more effective and flexible in studying the spectral analysis of boundary value problem than using the classical methods in operator theory. The goal of this paper is to study Prüfer approach to spectral analysis of periodic Sturm–Liouville problem with transmission condition. Since we are dealing with a singular problem, the characteristic function

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we obtained is a piecewise function. At the end of the study, the existence of eigenvalues of investigated problem by using Prüfer transformation is given.

Keywords: Sturm–Liouville Operator; Prüfer Transformation; Floquet Theory; Transmission Condition; Eigenvalues.

1. INTRODUCTION

In physics many problems arise in the form of Sturm–Liouville equation

$$-\left(py'\right)' + qy = \lambda wy, \tag{1}$$

where λ is the spectral parameter and the function w, sometimes denoted r, is called the weight or density function. Throughout this study, it will be assumed that $p(x),\ w(x) > 0,\ \frac{1}{p(x)},\ q(x),\ w(x) \in L_1(J)$ and $J = (0, c) \cup (c, \pi)$. In the literature, most of the time such equations have been studied with the boundary conditions of the form

$$\begin{aligned} y(0)\cos\alpha &= (py)'(0)\sin\alpha, \\ y(\pi)\cos\beta &= (py)'(\pi)\sin\beta. \end{aligned} \tag{2}$$

In this paper, we shall study systematically the Sturm–Liouville equation (1) in which the solutions have a discontinuity in an interior point. More precisely, we consider the boundary value problem (1) on J with periodic (antiperiodic) boundary conditions

$$\begin{aligned} y(0) &= \pm\, y(\pi), \\ y'(0) &= \pm\, y'(\pi) \end{aligned} \tag{3}$$

and with the jump (or transmission) conditions

$$\begin{aligned} y(c+0) &= y(c-0), \\ y'(c+0) &= y'(c-0), \end{aligned} \tag{4}$$

where c is an inner discontinuity point. The studies on periodic Sturm–Liouville operators are a great part of the spectral analysis of Sturm–Liouville operators and they have many applications in mathematics and applied sciences. Sturm–Liouville problems with periodic boundary conditions are also common in physics, such as quantum physics, the motion of a particle in the bulk matter, frequency and vibration theory.[1]

The spectral analysis of periodic Sturm–Liouville operators is based on the idea of Floquet/Hill method,[2,3] which uses the arguments of operator theory, but requires seriously background in functional analysis. Such a method has been used in many works and we refer the readers to some works for more details.[2–12] Other works based on spectral analysis of Sturm–Liouville operator with transmission condition can be found in the literature.[13–15]

Prüfer transformation provides an alternative or sometimes more convenient way of describing an investigated problem as it will be in our study. It was observed very early by Prüfer that bears his name.[16] Then the case where p, q and w are continuous in (1) has been analyzed in Ref. 17. Although the tools of Prüfer transformation have been available and applicable to various fields of study, the use of it in spectral analysis of Sturm–Liouville problems has only been very limited.[18] After a long time, Binding and his friends published a series of papers. The first paper related to investigation of asymptotic for Atkinson's semi-definite Sturm–Liouville eigenvalue problem by Prüfer angle was published in 2005.[19] Then again Binding *et al.* applied Prüfer angle approach to singular Sturm–Liouville problems with Molčanov potentials,[20] to periodic Sturm–Liouville problem[21] and to semi-definite Sturm–Liouville problems with coupling boundary conditions.[22–24]

Prüfer angle approach and its some features will be covered and adapted for considered problem in the following section. For more details about this section, see Ref. 17.

2. ADAPTATION OF PRÜFER TRANSFORMATION TO PROBLEM

In this section, we review Prüfer approach to the spectral analysis of Sturm–Liouville problem. In the simplest sense, Prüfer transformation can be seen as a technique introducing the polar coordinates in the phase plane. For a nonzero solution y of (1), let us say

$$y = r\sin\theta, \quad py' = r\cos\theta. \tag{5}$$

It is well known from Ref. 17 that Eq. (1) for $y(x)$ can be transformed into the equivalent system of equation for r and θ

$$\theta' = \frac{\cos^2\theta}{p} + (\lambda w - q)\sin^2\theta, \tag{6}$$

$$r' = r\left(\frac{1}{p} + q - \lambda w\right)\sin\theta\cos\theta \tag{7}$$

with initial conditions

$$\theta(\lambda,\alpha,0) = \alpha, \quad r(\lambda,\alpha,0) = 1. \tag{8}$$

Here, $\alpha \in \mathbb{R}$, θ and r are the functions of (λ,α,x) and determined by initial value problem (6)–(8). When α is given, for conditions (2) the kth eigenvalue is determined by

$$\theta(\lambda,\alpha,\pi) = \beta + k\pi, \quad \alpha,\beta \in J,$$

where k is oscillation count of λ and also the number of zeros in J of any eigenfunction corresponding to λ. Now, we give some features of θ that we will need in the following section. First of all, $\theta(\lambda,\alpha,x)$ is an increasing function of x when $\theta = 0 \pmod{\pi}$. Next, $\theta(\lambda,\alpha,\pi)$ is monotone increasing and continuous function of λ. Finally, $\theta(\lambda,\alpha,\pi)\to\infty$ as $\lambda\to\infty$ and $\theta(\lambda,\alpha,\pi)\to 0$ as $\lambda\to-\infty$.

All arguments above suffice to the existence of unique $\lambda = \lambda_k(\alpha,\beta)$ with oscillation count k for each $k \geq 0$ (see Ref. 17).

Let us give some motivations and some facts to use the following section. First, $\theta(\lambda,\alpha,x)$ is C^1 and it follows from (6) and (8) that its derivative with respect to α, denoted by θ_α, satisfies the following variational initial value problem:

$$\theta'_\alpha = -2\theta_\alpha\left(\frac{1}{p} + q - \lambda w\right)\sin\theta\cos\theta, \qquad \theta_\alpha(\lambda,\alpha,0) = 1 \tag{9}$$

(see Refs. 4 and 6). It can be seen from (7) and (9) that $(r^2\theta_\alpha)' = 0$ and from (8), we obtain

$$r^2\theta_\alpha = 1. \tag{10}$$

At the end of this section, we give two corresponding equivalent conditions of (3) and (4) in the phase plane.

First, inserting (5) into (3) we find

$$\tan\theta(\lambda,\alpha,0) = \tan\theta(\lambda,\alpha,\pi),$$

now using the initial condition (8), we conclude that

$$\theta(\lambda,\alpha,\pi) = \alpha + k\pi,$$

where k is even for a periodic condition and k is odd for an antiperiodic condition. Here again, it follows from (3) that

$$r(\lambda,\alpha,\pi) = 1.$$

Now, from this equation and (10) we get

$$\theta_\alpha(\lambda,\alpha,\pi) = 1.$$

Second, it follows from transmission conditions (4) that

$$\begin{aligned} r(\lambda,\alpha,c+0)&\sin\theta(\lambda,\alpha,c+0)\\ &= r(\lambda,\alpha,c-0)\sin\theta(\lambda,\alpha,c-0),\\ r(\lambda,\alpha,c+0)&\cos\theta(\lambda,\alpha,c+0)\\ &= r(\lambda,\alpha,c-0)\cos\theta(\lambda,\alpha,c-0) \end{aligned}$$

hence

$$\tan\theta(\lambda,\alpha,c+0) = \tan\theta(\lambda,\alpha,c-0)$$

thus

$$\theta(\lambda,\alpha,c-0) = \theta(\lambda,\alpha,c+0) + k\pi.$$

From the above equality, we obtain three cases of $\theta(\lambda,\alpha,x)$ according to x,

$$\theta(\lambda,\alpha,x) = \begin{cases} \alpha & \text{if } x = 0,\\ \alpha + 2k\pi & \text{if } x \in (0,c),\\ \alpha + k\pi & \text{if } x \in (c,\pi]. \end{cases} \tag{11}$$

Hereafter, it will be more useful to study with the following difference:

$$\delta(\lambda,\alpha) = \begin{cases} \theta(\lambda,\alpha,c-0) - \theta(\lambda,\alpha,0) & \text{if } x \in [0,c),\\ \theta(\lambda,\alpha,\pi) - \theta(\lambda,\alpha,0) & \text{if } x \in (c,\pi]. \end{cases} \tag{12}$$

Hence, the eigenvalue conditions will be in the form as follows, which is called the characteristic function

$$\delta(\lambda,\alpha) = \begin{cases} 2k\pi & \text{if } x \in [0,c),\\ k\pi & \text{if } x \in (c,\pi] \end{cases} \quad \text{and} \quad \delta_\alpha(\lambda,\alpha) = 0. \tag{13}$$

It is easily seen from (11) that θ is periodic function with period π

$$\theta(\lambda,\alpha+\pi,x) = \theta(\lambda,\alpha,x) + \pi. \tag{14}$$

Therefore by using (12) and (13), we deduce that δ is also periodic function with period π since

$$\begin{aligned}
&\delta(\lambda, \alpha+\pi) \\
&= \begin{cases} \theta(\lambda, \alpha+\pi, c-0) \\ \quad -\theta(\lambda, \alpha+\pi, 0) & \text{if } x \in [0,c), \\ \theta(\lambda, \alpha+\pi, \pi) \\ \quad -\theta(\lambda, \alpha+\pi, 0) & \text{if } x \in (c,\pi] \end{cases} \\
&= \begin{cases} \theta(\lambda, \alpha, c-0) + \pi - (\alpha+\pi) & \text{if } x \in [0,c), \\ \theta(\lambda, \alpha, \pi) + \pi - (\alpha+\pi) & \text{if } x \in (c,\pi] \end{cases} \\
&= \begin{cases} \alpha + 2k\pi + \pi - \alpha - \pi & \text{if } x \in [0,c), \\ \alpha + k\pi + \pi - \alpha - \pi & \text{if } x \in (c,\pi] \end{cases} \\
&= \begin{cases} 2k\pi & \text{if } x \in [0,c), \\ k\pi & \text{if } x \in (c,\pi] \end{cases} \\
&= \delta(\lambda, \alpha).
\end{aligned}$$

The characteristic function $\delta(\lambda, \alpha)$, derived by (13) will play a critical role in the following section.

3. EXISTENCE OF EIGENVALUES

In this section, we give the main theorem. But before that, we analyze the extremum values of the function $\delta(\lambda, \alpha)$ by the following lemma. These extremum values will be very important for the considered problem. The equalities in (13) show us $\delta(\lambda, \alpha)$ has a critical point at $2k\pi$ in the interval $[0, c)$ and $k\pi$ in the interval $(c, \pi]$ with respect to α. For our problem, these critical points are also extreme points. We begin by extending the definition of $\delta(\lambda, \alpha)$ from $\alpha \in [0, c) \cup (c, \pi]$ to all real α-axes by using the periodicity of δ. Let us introduce the notations $\min(\lambda)$ and $\max(\lambda)$ as a minimum and a maximum of $\delta(\lambda, \alpha)$, respectively, when α is taking all over the real values.

Now, we adapt Lemma 3.1 in Ref. 21 related to some basic properties of $\min(\lambda)$ and $\max(\lambda)$ for considered problem. These properties will be used in the following main theorem.

Lemma 1. *The functions* $\min(\lambda)$ *and* $\max(\lambda)$ *have the following four properties*:

(1) *The functions* $\min(\lambda)$ *and* $\max(\lambda)$ *are continuous and non-decreasing.*

(2) *For each* λ, *the inequalities*

$$\min(\lambda) \le \max(\lambda) < \min(\lambda) + \pi$$

are valid.

(3)

$$\lim_{\lambda \to -\infty} \min(\lambda) = -\pi$$

and

$$\lim_{\lambda \to -\infty} \max(\lambda) = 0.$$

(4) $\lim_{\lambda \to \infty} \min(\lambda) = +\infty$.

Proof. (1) On differentiating Eq. (6) partially with respect to λ, we obtain

$$\theta'_\lambda = -2\theta_\lambda \left(\frac{1}{p} + q - \lambda w\right) \sin\theta \cos\theta + w \sin^2\theta,$$

and also

$$\theta_\lambda(\lambda, \alpha, 0) = 0.$$

Here, $\theta_\lambda \in C^1$ and satisfies the above variational problem. So,

$$|\theta_\lambda|' \le 2|\theta_\lambda| \left|\frac{1}{p} + q - \lambda w\right|,$$

which is independent of α.

(2) For any λ take α_0 such that $\min(\lambda, \alpha_0) = \delta(\lambda, \alpha_0)$. Now, let $\alpha \in (\alpha_0, c) \cup (c, \alpha_0 + \pi]$ if $\alpha_0 < c$ and $\alpha \in (\alpha, \alpha_0 + \pi]$ if $\alpha_0 > c$. For both cases, since $\alpha \le \alpha_0 + \pi$ and $\theta(\lambda, \alpha, \pi)$ is increasing in α, from (12) and (14), the following inequality is achieved:

$$\begin{aligned}
&\delta(\lambda, \alpha) \\
&= \begin{cases} \theta(\lambda, \alpha, c-0) - \alpha & \text{if } x \in [0,c), \\ \theta(\lambda, \alpha, \pi) - \alpha & \text{if } x \in (c,\pi] \end{cases} \\
&\le \begin{cases} \theta(\lambda, \alpha_0+\pi, c-0) - \alpha & \text{if } x \in [0,c), \\ \theta(\lambda, \alpha_0+\pi, \pi) - \alpha & \text{if } x \in (c,\pi] \end{cases} \\
&= \begin{cases} \theta(\lambda, \alpha_0, c-0) + \pi - \alpha \\ \quad + \alpha_0 - \alpha_0 & \text{if } x \in [0,c), \\ \theta(\lambda, \alpha_0, \pi) + \pi - \alpha \\ \quad + \alpha_0 - \alpha_0 & \text{if } x \in (c,\pi] \end{cases} \\
&= \pi + \alpha_0 - \alpha + \delta(\lambda, \alpha_0).
\end{aligned}$$

It follows from $\alpha_0 - \alpha < 0$ for both cases and we get

$$\delta(\lambda, \alpha) < \min(\lambda) + \pi.$$

When α is changing from α_0 to $\alpha_0 + \pi$, there will be at least one value of α such that its value under the function $\delta(\lambda, \alpha)$ is going to be $\max(\lambda)$. So, we obtain

$$\max(\lambda) < \min(\lambda) + \pi.$$

(3) From (12),

$$\delta(\lambda,0)=\begin{cases}\theta(\lambda,0,c-0) & \text{if } x\in[0,c),\\ \theta(\lambda,0,\pi) & \text{if } x\in(c,\pi]\end{cases}$$
$$=\begin{cases}2k\pi & \text{if } x\in[0,c),\\ k\pi & \text{if } x\in(c,\pi]\end{cases}$$

so it can be noted that

$$\max(\lambda)\geq\delta(\lambda,0)>0. \tag{15}$$

On the other hand, by using (12) for $\alpha\in[0,c)\cup(c,\pi]$

$$\delta(\lambda,\alpha)\geq-\alpha>-\pi$$

hence we obtain $\min(\lambda)>-\pi$. Therefore, the inequality

$$-\pi<\min(\lambda)\leq\delta(\lambda,\alpha)$$

is valid, where

$$\delta(\lambda,\alpha)=\begin{cases}\theta(\lambda,\alpha,c-0)-\alpha & \text{if } x\in[0,c),\\ \theta(\lambda,\alpha,\pi)-\alpha & \text{if } x\in(c,\pi].\end{cases}$$

As $\lambda\to-\infty$, since $\theta\to 0$ (see Sec. 2) we have $\min(\lambda)\to-\pi$. Finally, using the inequality (15) and taking the limit of inequality $0\leq\max(\lambda)<\min(\lambda)+\pi$ in (b) as $\lambda\to-\infty$, we get $\max(\lambda)\to 0$.

(4) We know from (15) that $\max(\lambda)\geq\delta(\lambda,0)$. If we take the limit of both sides of this inequality as $\lambda\to\infty$ and using the fact that $\theta(\lambda,\alpha,\pi)\to\infty$ as $\lambda\to\infty$ (see Sec. 2), from the comparison test we arrive at $\max(\lambda)\to\infty$. From this and (b) we conclude that $\min(\lambda)\to\infty$ as $\lambda\to\infty$. □

Now, the main theorem regarding the existence of the eigenvalues of the studied problem can be stated as follows.

Theorem 2. *For each* $\lambda\in\mathbb{R}$ *the function* $\delta(\lambda,\alpha)$ *has no critical values except only* $\min(\lambda)$ *and* $\max(\lambda)$.

Proof. We will prove the theorem by contradiction and we start by assuming the opposite that for some λ, $\delta(\lambda,\alpha)$ has three distinct critical values δ_j corresponding to $\alpha_j\in[0,c)\cup(c,\pi]$ for $j=0,1,2$. Here, we can choose them in order $0\leq\alpha_0<\alpha_1<\alpha_2<\pi$. Let a function $y_j(x)$ be the solution of (1) with Prüfer angle $\theta(\lambda,\alpha_j,x)$ and Prüfer radius $r(\lambda,\alpha_j,x)$ according to (5). For the sake of simplicity we will use the abbreviation $\theta_j=\theta(\lambda,\alpha_j,x)$ for $j=0,1,2$.

The Wronskian of the solutions $y_0(x)$ and $y_1(x)$ under the transformation (5) is

$$\begin{aligned}W(y_0,y_1)&=p(x)[y_0'(x)y_1(x)-y_0(x)y_1'(x)]\\&=p(x)\left[\frac{r(\lambda,\alpha_0,x)\cos\theta(\lambda,\alpha_0,x)}{p(x)}\right.\\&\quad\times r(\lambda,\alpha_1,x)\sin\theta(\lambda,\alpha_1,x)\\&\quad-r(\lambda,\alpha_0,x)\sin\theta(\lambda,\alpha_0,x)\\&\quad\left.\times\frac{r(\lambda,\alpha_1,x)\cos\theta(\lambda,\alpha_1,x)}{p(x)}\right]\\&=r(\lambda,\alpha_0,x)r(\lambda,\alpha_1,x)\sin(\theta(\lambda,\alpha_1,x).\\&\quad-\theta(\lambda,\alpha_0,x)).\end{aligned}$$

The above equality indicates that $W(y_0,y_1)$ is independent of x. Therefore, the values of Wronskian at $x=0$ and $x=\pi$ must be equal

$$W(y_0(\pi),y_1(\pi))=W(y_0(0),y_1(0))$$

and so

$$\begin{aligned}&\sin(\theta(\lambda,\alpha_1,\pi)-\theta(\lambda,\alpha_0,\pi))\\&\quad=\sin(\theta(\lambda,\alpha_1,0)-\theta(\lambda,\alpha_0,0)).\end{aligned}$$

Here, we used the fact that

$$r(\lambda,\alpha_j,0)=r(\lambda,\alpha_1,0)=1.$$

Now, from the abbreviation given above and (8)

$$\sin(\theta_1-\theta_0)=\sin(\alpha_1-\alpha_0). \tag{16}$$

Since $\alpha_0<\alpha_1<\alpha_0+\pi$ by using the fact that θ is increasing function and from (14) we have the following inequality:

$$\begin{aligned}\theta(\lambda,\alpha_0,x)<\theta(\lambda,\alpha_1,x)&<\theta(\lambda,\alpha_0+\pi,x)\\&=\theta(\lambda,\alpha_0,x)+\pi.\end{aligned}$$

Hence, we get

$$\theta_0<\theta_1<\theta_0+\pi.$$

Since $\delta_0=\delta_1$, $\theta_0-\alpha_0\neq\theta_1-\alpha_1$ so (16) implies

$$\theta_1-\theta_0+\alpha_1-\alpha_0=\pi. \tag{17}$$

Similarly, by taking the solution pairs $y_1(x)$, $y_2(x)$ and $y_0(x)$, $y_2(x)$ and repeating same operations we obtain two more equalities, respectively,

$$\theta_2-\theta_1+\alpha_2-\alpha_1=\pi \tag{18}$$

and

$$\theta_0-\theta_2+\alpha_0-\alpha_2=-\pi. \tag{19}$$

The sum of the equalities (17)–(19) side by side implies contradiction that 0 is equal to π. □

The following corollary gives the relation between the eigenvalues of the boundary value problem and the critical values of the function $\delta(\lambda,\alpha)$.

Corollary 3. *The critical values of the function $\delta(\lambda, \alpha)$ correspond to all the eigenvalues of the boundary value problem* (1)–(4).

4. CONCLUSION

In this work, we have extended the spectral analysis of Sturm–Liouville problem by Prüfer transformation to the periodic singular Sturm–Liouville problem with piecewise characteristic. It should be noted that the spectral analysis using the Prüfer transformation is simpler and clearer than classical methods. Here, we have analyzed the existence of the eigenvalues of the boundary value problem given with transmission condition that we have considered. The critical point here is that the characteristic function, which is very important for the main theorem, appears as a piecewise function due to the given transmission condition. Therefore, we proved the lemma given by Binding and Volkmer[21] again according to the new characteristic function. Then, at the end of the paper, we gave the main theorem showing the existence of the eigenvalues of the problem.

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