



# Analysis of Generating Functions for Special Words and Numbers and Algorithms for Computation

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**Abstract.** Our aim is to construct and compute efficient generating functions enumerating the  $k$ -ary Lyndon words having prime number length which arise in many branches of mathematics and computer science. We prove that these generating functions coincide with the Apostol–Bernoulli numbers and their interpolation functions and obtain other forms of these generating functions including not only the Frobenius–Euler numbers, but also the Fubini type numbers. Moreover, we derive some identities, relations and combinatorial sums including the numbers of the  $k$ -ary Lyndon words, the Bernoulli numbers and polynomials, the Stirling numbers and falling factorials. Using these generating functions and recurrence relation for the Apostol–Bernoulli numbers, we give two algorithms to compute these generating functions. Using these algorithms, we compute some infinite series formulas including the number of the  $k$ -ary Lyndon words on some special classes of primes with the purpose of providing some numerical evaluations about these generating functions. In addition, we approximate these generating functions by the rational functions of the Apostol–Bernoulli numbers to show that the complexity of the aforementioned algorithms may be decreased by means of approximation method which are illustrated by some numerical evaluations with their plots for varying prime numbers. Finally, using Bell polynomials (i.e., exponential functions) approach to the numbers of Lyndon words, we construct the exponential generating functions for the numbers of Lyndon words. Finally, we define a new family of special numbers related to these special words and investigate some of their fundamental properties.

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## 1. Introduction

In 1954, famous mathematician Lyndon [27] defined so-called Lyndon words which arise in many areas of mathematics, computer science and biological sciences. Their potential usage in mathematical modelling have made these words even more attractive to the researchers. Especially, in algebra and algebraic combinatorics there exists various papers on some formulas correspond not only the numbers of these words, but also dimensions related to free monoid theory and free lie algebra. Moreover, the subject of algorithmic complexity and algorithm desing for generating these words have been studied by many authors in recent years (cf. [6, 10, 11, 16–18, 22, 26–28, 33, 52]).

For a more detailed understanding about the Lyndon words, we need the following definitions and brief preliminary informations about these special words.

The Lyndon word is a lexicographically smallest in its conjugate class which is a set formed by cyclically shifting of the letters in the word. These words are called  $k$ -ary Lyndon word of length  $n$  if it is derived from the  $k$ -letter finite lexicographically ordered alphabet  $\Sigma$  and it has  $n$  digits (cf. [26]). Here, lexicographical order means a dictionary order (cf. [36]; and see also the references cited therein). It is well-known that primitive combinatorial necklaces have recently been studied as representative of these words (see [6, 28]; and also the references cited therein). Therefore, we recall some information about the combinatorial necklace and their periodicity and primitiveness concepts. In the work of Metropolis and Rota [28], a combinatorial necklace is defined as the result of placing  $n$  colored beads around a circle when given a set of  $k$  distinct colours. Figure 1 provides an example of combinatorial necklace for  $k = 2$  and  $n = 8$ , that is, the following figure describes how binary words are represented by a combinatorial necklace with 8 colored beads.

In Fig. 1, blue and red beads are represented by letter 0 and letter 1, respectively. Then, the rotations on the combinatorial necklace are representing 8 different words given by the following set:

$$\{10101101, 01011011, 10110110, \\ 01101101, 11011010, 10110101, 01101011, 11010110\}.$$

Observe that all words in the above list are equivalent according to rotation process. One can easily see that these words are generated by the combinatorial necklace in Fig. 1 and they are belongs to the set of all conjugates (rotations) of a given word above. Hence, the aforementioned combinatorial necklace is called conjugacy class of all strings above. If representative word of a combinatorial necklace can be expressed by a positive power of a shorter word, then the combinatorial necklace is called periodic necklace. Otherwise, the combinatorial necklace is called aperiodic(primitive) necklace. According to work of Cusick and Stanica [14, p. 36], *a  $k$ -ary necklace is an equivalence class of  $k$ -ary words under rotation. As a representative of such an equivalence class one takes the smallest in the lexicographical order (as in dictionary).* That is, it may be said that aperiodic necklace class representative with  $n$  digits is called a  $k$ -ary Lyndon word of length  $n$  over the

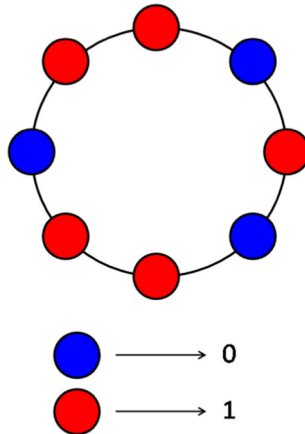


Figure 1. A combinatorial necklace

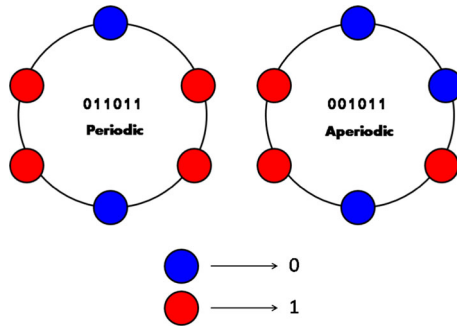


Figure 2. Examples for the periodic and aperiodic necklaces with their representative words

$k$ -letter alphabet  $\Sigma$ . For detail information about properties of the aperiodic (primitive) words, readers may consult [6,28].

Figure 2 shows an example of periodic and aperiodic necklaces with their representative words. Observe that representative word of the left necklace,  $011011 = (011)^2$  can be expressed a positive power of a shorter word. Therefore, the left necklace and its representative word  $011011$  are called periodic. But, on the right necklace is aperiodic necklace, because its representative word  $001011$  can not be written in terms of a shorter word. Thus, it is clear that  $001011$  is a 2-ary Lyndon word of length 6.

In addition, by replacing 6 colored beads in different combination obtained all 2-ary Lyndon words of length 6, over the lexicographically ordered 2-letter alphabet  $\Sigma = \{0, 1\}$ , are given as follows:

$\{000001, 000011, 000101, 000111, 001011, 001101, 001111, 010111, 011111\}$ .

For positive integer  $n$ , the lexicographical concatenation of Lyndon words of length dividing  $n$  generates a de Bruijn sequence of order  $n$  which

Table 1. Some numerical values of  $L_k(n)$

		$k$								
		1	2	3	4	5	6	7	8	9
$n$	1	1	2	3	4	5	6	7	8	9
	2	0	1	3	6	10	15	21	28	36
	3	0	2	8	20	40	70	112	168	240
	4	0	3	18	60	150	315	588	1008	1620
	5	0	6	48	204	624	1554	3360	6552	11808

is defined as a cyclic sequence of length  $k^n$  in which each probable word of length  $n$  over the  $k$ -letter alphabet appears consecutively as a subword exactly once. The number of  $k$ -ary Lyndon words having length  $n$ , denoted by  $L_k(n)$ , is given by the following explicit formula (cf. [6, 14, 26, 36]):

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) k^d, \quad (k, n \in \mathbb{Z}^+) \tag{1.1}$$

so that the sum is over all positive divisors of  $n$  and  $\mu$  denotes the Möbius function defined by (cf. [2]):

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes,} \\ 0 & \text{if } n \text{ is not square-free.} \end{cases}$$

Using Eq. (1.1), we compute a few values of the number of  $k$ -ary Lyndon words of length  $n$ ,  $L_k(n)$  for  $n = 1, 2, 3, 4, 5$  and  $k = 1, 2, 3, 4, 5, 6, 7, 8, 9$  as seen in Table 1.

For example, from Table 1, one may see that the number of 3-ary Lyndon words of length 4 is 18.

*Remark 1.1.* In addition to counting the  $k$ -ary Lyndon words of length  $n$ , Eq. (1.1) is well-known as Witt’s formula, which is used to count the number of monic irreducible polynomials of degree  $n$  over Galois field with  $k$  elements. We observe that there are connections between the aforementioned formula and dimension formula for the homogeneous subspaces of the free Lie algebra and the rank of the free abelian quotient. Furthermore, Witt’s formula is also called necklace polynomial when used to count the number of  $k$ -ary aperiodic (primitive) necklaces of length  $n$  (cf. [10, 22, 25, 27, 33, 52]).

Using (1.1), we can easily get the following explicit formula for the numbers of the  $k$ -ary Lyndon words of length prime-power  $p^m$ ,  $L_k(p^m)$ :

$$L_k(p^m) = \frac{k^{p^{m-1}}(k^{p^{m-1}(p-1)} - 1)}{p^m}, \tag{1.2}$$

where  $p$  is a prime number and  $m$  is a positive integer.

Recently, generating function methods are used in almost all branches of mathematics, mathematical physics and other related areas. Therefore, because of power of generating functions, the main motivation and purpose of this paper is to construct and compute efficient generating functions enumerating the  $k$ -ary Lyndon words having prime number length and give numerical evaluations for these functions with their graphs.

Throughout this paper, let  $p$  denote any prime number. Then, in [25], we defined the generating functions for the numbers of the  $k$ -ary Lyndon words of length  $p$  as follows

$$f_{L_B}(t, p) = \sum_{k=1}^{\infty} L_k(p)t^k, \tag{1.3}$$

where  $|t| < 1$ .

Note that there is one generating function for each value of  $p$ .

In this paper, we provide an explicit formula for these generating functions including the Apostol–Bernoulli numbers  $\mathcal{B}_n(t)$  are given by the following theorem which is proved in Sect. 2.

**Theorem 1.1.** *Let  $p$  be a prime number. Then*

$$f_{L_B}(t, p) = \frac{\mathcal{B}_2(t)}{2p} - \frac{\mathcal{B}_{p+1}(t)}{p(p+1)}, \tag{1.4}$$

where  $\mathcal{B}_p(t)$  denotes the Apostol–Bernoulli numbers which are rational functions of parameter  $t$ .

We summarize our paper as follows:

In Sect. 2, we recall some definitions and identities related to the Apostol–Bernoulli numbers and polynomials and the unified zeta-type functions. Using these numbers and functions, we prove Theorem 1.1. Moreover, we introduce other forms of the generating functions  $f_{L_B}(t, p)$ , including not only the Frobenius–Euler numbers, but also the Fubini type numbers.

In Sect. 3, we derive some identities, relations and combinatorial sums including the numbers of the  $k$ -ary Lyndon words, the Bernoulli numbers and polynomials, the Stirling numbers and falling factorials.

In Sect. 4, we give an efficient computation algorithm for Apostol–Bernoulli numbers, as well as the generating functions of the number of  $k$ -ary Lyndon words with prime number length. By applying these algorithms, we give some numerical computations of the generating functions on the some special classes of primes. We also give some observations and remarks on our computations and also classes of prime numbers.

In Sect. 5, we plot graphics of the generating functions  $f_{L_B}(t, p)$  for varying prime numbers to demonstrate the effects of prime numbers on the shape of the curve. We also present a weighed approximation of the generating functions  $f_{L_B}(t, p)$  by rational functions and also plot their graphics for certain values of prime numbers.

In Sect. 6, using Bell polynomials (i.e., exponential functions) approach to the numbers of Lyndon words, we construct the exponential generating functions for the numbers of Lyndon words. We define a new family of special

numbers related to these special words, and we investigate their fundamental properties.

## 2. Relation Between the Functions $f_{L_B}(t, p)$ and Interpolation Functions of the Apostol–Bernoulli Numbers

In this section, we prove a relation between the generating functions  $f_{L_B}(t, p)$  and the Apostol–Bernoulli numbers by using their interpolation functions reduced from the unified zeta-type functions. With the aid of some relations between the Apostol–Bernoulli numbers, the Frobenius–Euler numbers and a new family of numbers related to the Fubini type numbers, we also construct other forms of the generating functions  $f_{L_B}(t, p)$  related to not only the Frobenius–Euler numbers, but also the Fubini type numbers.

To give our results in this section, we recall some definitions and identities related to the Apostol–Bernoulli numbers and the unified zeta-type functions:

We firstly recall the Apostol–Bernoulli polynomials  $\mathcal{B}_k(x, t)$  defined by means of the following generating function [1]

$$\frac{ze^{zx}}{te^z - 1} = \sum_{k=0}^{\infty} \mathcal{B}_k(x, t) \frac{z^k}{k!}, \tag{2.1}$$

where  $|z| < 2\pi$  when  $t = 1$  and  $|z| < |\log t|$  when  $t \neq 1$  and  $t \in \mathbb{C}$ . From the above generating function, we have

$$\mathcal{B}_m(x, t) = \sum_{j=0}^m \binom{m}{j} x^{m-j} \mathcal{B}_j(t), \tag{2.2}$$

where  $m$  is a positive integer.

$\mathcal{B}_m(t) = \mathcal{B}_m(0, t)$  denotes the Apostol–Bernoulli numbers which are defined by means of the following generating function

$$\frac{z}{te^z - 1} = \sum_{k=0}^{\infty} \mathcal{B}_k(t) \frac{z^k}{k!}, \tag{2.3}$$

(cf. [1, 8, 20, 32, 45, 48, 50, 51]; and the references cited therein).

Using the Umbral calculus convention in Eq. (2.3), one can easily have the following recurrence relation

$$\mathcal{B}_m(t) = \frac{t}{1-t} \sum_{j=0}^{m-1} \binom{m-1}{j} \mathcal{B}_j(t), \quad m \geq 2, \tag{2.4}$$

with  $\mathcal{B}_0(t) = 0$  and  $\mathcal{B}_1(t) = (t - 1)^{-1}$ .

Using this recurrence relation, one may compute the values of the Apostol–Bernoulli numbers,

$$\begin{aligned} \mathcal{B}_2(t) &= \frac{-2t}{(t-1)^2}, & \mathcal{B}_3(t) &= \frac{3t(t+1)}{(t-1)^3}, & \mathcal{B}_4(t) &= \frac{-4t(t^2+4t+1)}{(t-1)^4}, \\ \mathcal{B}_5(t) &= \frac{5t(t^3+11t^2+11t+1)}{(t-1)^5}, & \mathcal{B}_6(t) &= \frac{-6t(t^4+26t^3+66t^2+26t+1)}{(t-1)^6}, \end{aligned}$$

$$\mathcal{B}_7(t) = \frac{7t(t^5 + 57t^4 + 302t^3 + 302t^2 + 57t + 1)}{(t - 1)^7},$$

etc.

Using  $\varphi$ -polynomials introduced in [30] by

$$\varphi_k(t) = (1 - t)^k + t \sum_{\nu=1}^k \binom{k+1}{\nu} (1 - t)^{\nu-1} \varphi_{k-\nu}(t), \quad k \geq 0, \tag{2.5}$$

where the sum on the right hand side in (2.5) is empty for  $k = 0$ , the Apostol-Bernoulli numbers can be expressed in the form

$$\mathcal{B}_m(t) = (-1)^{m-1} m \frac{t\varphi_{m-2}(t)}{(t - 1)^m}, \quad m \geq 1. \tag{2.6}$$

As we mentioned in [30], the polynomials  $\varphi_k(t)$  of degree  $k$  are monic,  $\varphi_k(0) = 1$ ,  $\varphi_k(1) = (k + 1)!$ , and they are self-inversive, i.e.,  $t^k \varphi_k(1/t) \equiv \varphi_k(t)$  (cf. [29, pp. 16–18]). Also, an explicit determinant form of  $\varphi_k(t)$  has been given in [30].

From the above expressions of the Apostol–Bernoulli numbers  $\mathcal{B}_m(t)$ , we see that they are rational functions of the parameter  $t$ , with a pole of the order  $m$  at  $t = 1$ .

Secondly, we also recall the generating functions for the unification of the Bernoulli, Euler and Genocchi polynomials of higher-order,  $\mathcal{Y}_{n,t}^{(\nu)}(x; k, a, b)$ , which has been recently defined by Aygunes and Simsek [3] for  $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$  ( $\mathbb{N} := \{1, 2, 3, \dots\}$ );  $a, b \in \mathbb{R}^+$ ;  $t \in \mathbb{C}$ :

$$\left( \frac{2^{1-k} z^k}{t^b e^z - a^b} \right)^\nu e^{zx} = \sum_{n=0}^\infty \mathcal{Y}_{n,t}^{(\nu)}(x; k, a, b) \frac{z^n}{n!}, \tag{2.7}$$

where  $x \in \mathbb{R}$ . Note that for  $\nu = 1$  the above functions reduce to the unification of the Bernoulli, Euler and Genocchi polynomials,  $\mathcal{Y}_{n,t}^{(1)}(x; k, a, b)$ , which has been given by Ozden et al. [31]. Also, note that

$$\mathcal{Y}_{n,t}^{(1)}(k, a, b) = \mathcal{Y}_{n,t}^{(1)}(0; k, a, b)$$

(cf. [3, 31]).

For  $k, b \in \mathbb{N}_0$  and  $\nu \in \mathbb{N}$ , the unification of the multiple Lerch-zeta type functions  $\zeta_t^{(\nu)}(s, k, a, b)$ , which also has been recently defined by Aygunes and Simsek [3, p. 370, Definition 1] given as follows:

$$\zeta_t^{(\nu)}(s, k, a, b) = a^{-bv} \left( -\frac{1}{2} \right)^{(k-1)\nu} \sum_{n=1}^\infty \binom{n + \nu - 1}{n} \frac{1}{n^s} \left( \frac{t}{a} \right)^{bn}, \tag{2.8}$$

where  $s \in \mathbb{C}$  when  $|t/a| < 1$  and  $\text{Re}(s) > 1$  when  $|t/a| = 1$  and note that the function  $\zeta_t^{(\nu)}(s, k, a, b)$  denotes the interpolation functions for the unification of the Bernoulli, Euler and Genocchi numbers of higher orders at negative integers (cf. [3]).

*Remark 2.1.* Substituting special value of  $a, b$  and  $k$  into (2.8), we have the family of zeta functions such as the Hurwitz-Lerch zeta functions, the Hurwitz zeta functions, the Riemann zeta functions and the others. For the family of zeta functions, see following studies [1, 3, 24, 31, 32, 37–45, 47–49, 51].

Aygunes and Simsek [3, p. 371, Theorem 1] also proved the following relation for  $r > k\nu$ :

$$\zeta_t^{(\nu)}(k\nu - r, k, a, b) = (-1)^{k\nu} \frac{\mathcal{Y}_{r,t}^{(\nu)}(k, a, b)}{\binom{r}{k\nu} (k\nu)!}. \tag{2.9}$$

In the light of the above preliminaries, we give the following proof for Theorem 1.1:

*Proof of Theorem 1.1.* Substituting  $s = -m$  and  $k = a = b = \nu = 1$  into (2.8), we get

$$\zeta_t^{(1)}(-m, 1, 1, 1) = \sum_{n=1}^{\infty} t^n n^m. \tag{2.10}$$

Also substituting  $r = 1 + m$  and  $k = a = b = \nu = 1$  into (2.9), we obtain

$$\zeta_t^{(1)}(-m, 1, 1, 1) = -\frac{\mathcal{Y}_{1+m,t}^{(1)}(1, 1, 1)}{1 + m}. \tag{2.11}$$

From (2.1) and (2.7), one can easily get

$$\mathcal{Y}_{n,t}^{(1)}(1, 1, 1) = \mathcal{B}_n(t). \tag{2.12}$$

That is, in [3,31], we see that the functions  $\zeta_t^{(1)}(-m, 1, 1, 1)$  interpolates the Apostol–Bernoulli numbers by Eq. (2.11).

Therefore, by combining (2.10), (2.11) and (2.12) with (1.2) and (1.3), we arrive at desired result.  $\square$

*Remark 2.2.* For  $p = 2, 3, 5, 7, 11, \dots$ , the generating functions  $f_{LB}(t, p)$  are

$$\begin{aligned} f_{LB}(t, 2) &= \frac{t^2}{(1-t)^3}, \\ f_{LB}(t, 3) &= \frac{2t^2}{(1-t)^4}, \\ f_{LB}(t, 5) &= \frac{6t^2}{(1-t)^6} (t^2 + 2t + 1), \\ f_{LB}(t, 7) &= \frac{6t^2}{(1-t)^8} (3t^4 + 28t^3 + 58t^2 + 28t + 3), \\ f_{LB}(t, 11) &= \frac{6t^2}{(1-t)^{12}} (31t^8 + 2312t^7 \\ &\quad + 33388t^6 + 147544t^5 + 238250t^4 \\ &\quad + 147544t^3 + 33388t^2 + 2312t + 31), \\ f_{LB}(t, 13) &= \frac{210t^2}{(1-t)^{14}} (3t^{10} + 542t^9 + 16679t^8 \\ &\quad + 155048t^7 + 551510t^6 + 833396t^5 \\ &\quad + 551510t^4 + 155048t^3 + 16679t^2 + 542t + 3), \\ &\text{etc.} \end{aligned}$$

By combining (2.4) and (2.6) with (1.4), we get the following result:

**Corollary 2.1.** *Let  $p$  be a prime number. Then*

$$f_{L_B}(t, p) = \frac{1}{p} \left( \frac{t}{1-t} \right)^2 \sum_{j=1}^{p-1} \binom{p}{j} \frac{\varphi_{j-1}(t)}{(1-t)^j}.$$

Alternatively,

$$f_{L_B}(t, p) = -\frac{t}{p(t-1)} \left( \frac{1}{t-1} - \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} \mathcal{B}_j(t) \right).$$

**2.1. Other Forms of the Generating Functions  $f_{L_B}(t, p)$**

Here, we modify Eq. (1.4) using the following relationship between the Apostol–Bernoulli numbers and the Frobenius–Euler numbers:

$$\mathcal{B}_n(t) = \frac{n}{t-1} H_{n-1} \left( \frac{1}{t} \right), \tag{2.13}$$

where  $n \geq 1$  and  $H_n(t)$  denotes the Frobenius–Euler numbers (cf. [24, 41]) which are defined by means of the following generating function:

$$\frac{1-t}{e^z - t} = \sum_{n=0}^{\infty} H_n(t) \frac{z^n}{n!},$$

(cf. [21, 23, 37, 38, 43, 44]; and the references cited therein).

By combining Theorem 1.1 with Eq. (2.13), we obtain another form of the generating functions  $f_{L_B}(t, p)$ :

**Theorem 2.1.** *Let  $p$  be a prime number. Then*

$$f_{L_B}(t, p) = \frac{1}{p(t-1)} \left[ H_1 \left( \frac{1}{t} \right) - H_p \left( \frac{1}{t} \right) \right]. \tag{2.14}$$

Here, we also give a further form of the generating functions  $f_{L_B}(t, p)$  using relations between the Frobenius–Euler numbers and the Fubini type numbers.

In [39, Eq. (37), p. 21], Simsek defined a new family of numbers  $Y_n(u; a)$  by means of the following generating functions:

$$\frac{1}{a^t - u} = \sum_{n=0}^{\infty} Y_n(u; a) \frac{t^n}{n!}, \tag{2.15}$$

where  $a \in \mathbb{R}^+$  with  $a \geq 1$  and  $u \in \mathbb{C} \setminus \{1\}$ .

Using (2.15), Simsek [42] proved the following recurrence relation

$$Y_n(u; a) = \frac{1}{u-1} \sum_{j=0}^{n-1} \binom{n}{j} Y_j(u; a) (\log a)^{n-j}, \quad n \geq 1,$$

with  $Y_0(u; a) = 1/(1-u)$ , as well as a relationship between the Frobenius–Euler numbers and the numbers  $Y_n(u; a)$  in the form

$$H_n(u) = -\sum_{j=0}^{n-1} \binom{n}{j} Y_j(u; e), \quad n \geq 1. \tag{2.16}$$

Substituting  $u = 2$  and  $a = e$  into (2.15), we find the Fubini numbers  $w(n)$  as (cf. [19])

$$w(n) = -Y_n(2; e),$$

which count ordered partitions of nonempty subsets of the set  $\{1, 2, \dots, n\}$  (cf. [13, p. 228]) and these numbers are also related to combination lock and the de Bruijn sequences.

Thus, in the light of the above definitions, we give the following formulas related to the numbers  $Y_j(u; e)$ , the Fubini numbers and the generating functions  $f_{L_B}(t, p)$ .

Combining (2.14) and (2.16), we get the other generating function for  $L_k(p)$  as follows:

$$f_{L_B} \left( \frac{1}{u}, p \right) = -\frac{u}{p(1-u)^2} + \frac{u}{p(1-u)} \sum_{j=0}^{p-1} \binom{p}{j} Y_j(u; e).$$

Setting  $u = 2$  in the above equation, we get the following identity for the generating functions  $f_{L_B}(t, p)$  including the Fubini numbers

$$f_{L_B} \left( \frac{1}{2}, p \right) = \frac{2}{p} \left( -1 + \sum_{j=0}^{p-1} \binom{p}{j} w(j) \right).$$

### 3. Further Identities Related to the Bernoulli and Stirling Numbers

There are various applications of the Bernoulli numbers and polynomials in almost all branches of Mathematics. One of them is the sum of powers of consecutive integers including the Bernoulli numbers and polynomials. Using this sum, we first derive an identity including the numbers  $L_k(n)$ , Bernoulli numbers and polynomials. Secondly, using this identity, we obtain combinatorial sums. Moreover, using some relations related to the Stirling numbers and falling factorials, we also obtain relations for the numbers  $L_k(n)$ .

Now, we recall definition of the Bernoulli polynomials  $B_n(x)$  which are defined by means of the following generating function:

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{3.1}$$

where  $|t| < 2\pi$  and also  $B_n = B_n(0)$ , which denotes the Bernoulli numbers (cf. [5–47]; and the references cited therein). For the identities, relations and formulas related to the Bernoulli numbers, the readers may consult [46].

The sum of powers of consecutive integers including the Bernoulli numbers and polynomials is given by

$$\sum_{k=0}^m k^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1} \tag{3.2}$$

(cf. [5, p. 279], [15, 34, 47, 51]; see also the references cited therein).

By applying Mobius inversion formula to Eq. (1.1), we have

$$k^n = \sum_{d|n} dL_k(d). \tag{3.3}$$

Hence, summing Eq. (3.3) over all  $0 \leq k \leq m$  and combining with Eq. (3.2), we obtain the following theorem:

**Theorem 3.1.** *Let  $n \geq 1$ . Then*

$$\sum_{d|n} d \sum_{k=0}^m L_k(d) = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}. \tag{3.4}$$

Using (3.1), the Eq. (3.4) is reduced to the following combinatorial identity:

$$\sum_{d|n} d \sum_{k=0}^m L_k(d) = \frac{1}{n+1} \sum_{j=0}^n \binom{n+1}{j} (m+1)^{n+1-j} B_j.$$

Therefore, using Remark 5 in [40] with the definition of the second kind Stirling numbers, we arrive at the following combinatorial sums:

**Theorem 3.2.** *We have*

$$\sum_{d|n} d \sum_{k=0}^m L_k(d) = \frac{1}{n+1} \sum_{j=0}^n \sum_{\ell=0}^j \sum_{c=0}^{\ell} (-1)^{\ell+c} \binom{\ell}{c} \binom{n+1}{j} \frac{(\ell-c)^j}{\ell+1} (m+1)^{n+1-j}.$$

Observe that the above theorem gives us a way for direct computation of the sums

$$\sum_{d|n} d \sum_{k=0}^m L_k(d),$$

without help of any special numbers or polynomials.

To give our results related to the Stirling numbers, it is time to recall some definitions related to these numbers:

In [12, p. 202], a relation for the Stirling numbers of the second kind  $S(n, k)$ , including falling factorial  $(x)_n$ , is given by

$$\sum_{k=0}^n S(n, k)(x)_k = x^n, \tag{3.5}$$

where  $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$  and the generating function for these numbers is given by

$$\sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k.$$

The recurrence relation for the Stirling numbers of the second kind are given by

$$S(n, k) = S(n-1, k-1) + kS(n-1, k) \tag{3.6}$$

and  $S(0, 0) = 1$ ,  $S(n, k) = 0$  if  $k > n$ ;  $S(n, 0) = 0$  if  $n > 0$ . Furthermore, the falling factorial can be written in terms of the Stirling numbers of the first kind  $s(n, k)$  as

$$(x)_n = \sum_{k=0}^n s(n, k)x^k \tag{3.7}$$

(cf. [7, 12, 13, 51]; and the references cited therein).

Well-known combinatorial interpretation of the Stirling numbers of the second kind  $S(n, k)$  is that these numbers count the number of partitions of a finite set of  $n$  elements into  $k$  subsets (cf. [12]) and note that the absolute value of the Stirling numbers of the first kind,  $|s(n, k)|$  counts  $n$ -permutations with  $k$  cycles (cf. [7, 12]; see also the references cited therein).

Substituting (3.5) into left-hand side of (3.3), we have  $\sum_{d|n} dL_k(d) = \sum_{j=0}^n S(n, j)(k)_j$  and combining the above equation with (3.7), we get the following result:

**Theorem 3.3.** *We have*

$$\sum_{d|n} dL_k(d) = \sum_{j=0}^n \sum_{\ell=0}^j s(j, \ell)S(n, j)k^\ell. \tag{3.8}$$

If we replace  $k^n$  in (1.1) by (3.5) and combining it with (3.7), we get the following relations between the numbers  $L_k(n)$ , the Stirling numbers of the first kind and second kind:

**Corollary 3.1.** *We have*

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{j=0}^n S(n, j)(k)_j$$

and

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{j=0}^d \sum_{\ell=0}^j s(j, \ell)S(d, j)k^\ell. \tag{3.9}$$

*Remark 3.1.* Observe that Eq. (3.9) is the Mobius inversion of Eq. (3.8).

### 4. Computation Algorithms

In this section, using (2.4) and combining (2.4) with (1.4), we derive two efficient computation algorithms. In the next section, by applying these algorithms, we give some numerical applications related to the generating functions  $f_{L_B}(t, p)$ .

Firstly, we give the computation algorithm for the Apostol–Bernoulli numbers:

**Algorithm 1** Let  $m$  be a nonnegative integer and  $t \in \mathbb{C} \setminus \{1\}$ . This algorithm will return the  $m$ -th Apostol–Bernoulli number  $\mathcal{B}_m(t)$ , using recurrence relation given in Eq. (2.4), recursively.

```

procedure APOST_BERN_NUM( $m$ : nonnegative integer,  $t$ )
  Begin
  Global variable  $j$  : integer
  if  $m = 0$  then
    return 0
  else
    if  $m = 1$  then
      return  $1/(t - 1)$ 
    else
      return  $(t/(1 - t)) * \sum(Binomial\_Coef(m, j) * APOST\_BERN\_NUM(m - j, t), j, 1, m)$ 
    end if
  end if
end procedure

```

Now, we give a computation algorithm for the generating functions  $f_{L_B}(t, p)$  related to Apostol–Bernoulli numbers.

**Algorithm 2** Let  $p$  be a prime number,  $t \in \mathbb{C} \setminus \{1\}$  and  $|t| < 1$ . This algorithm will return  $f_{L_B}(t, p)$ , the generating function for the numbers of the  $k$ -ary Lyndon words of prime length  $p$  in (1.4), using APOST\_BERN\_NUM procedure given in the Algorithm 1.

```

procedure GEN_FUNC_LYNDON( $t, p$ : prime)
  Begin
  Local variables  $v_1, v_2 \leftarrow 0$ 
   $v_1 = (1/(2p)) * APOST\_BERN\_NUM(2, t)$ 
   $v_2 = (1/(p * (p + 1))) * APOST\_BERN\_NUM(p + 1, t)$ 
   $f_{L_B}(t, p) = v_1 - v_2$ 
  return  $f_{L_B}(t, p)$ 
end procedure

```

### 4.1. Numerical Computations of the Generating Function $f_{L_B}(t, p)$

Here, we give some numerical computations of the generating function  $f_{L_B}(t, p)$  on the some special classes of prime numbers.

Using Algorithms 1 and 2, we compute a few values of the generating functions  $f_{L_B}(t, p)$  for  $p = 2, 3, 5, 7$  and  $t = 1, 1/2, 1/3, -1/3, -1/4$  as follows

Observe that the generating functions are undefined in the column corresponding  $t = 1$ , since the generating functions  $f_{L_B}(t, p)$  have poles at  $t = 1$ .

There are many kind of prime numbers which are called classes of prime numbers generated by particular formula or having a special property such as Fermat primes, Mersenne primes, Fibonacci primes, Lucas primes and the others (cf. [53]). In Table 2, note that 3 and 7 are the Mersenne primes and 3 and 5 are the Fermat primes.

Table 2. Some numerical values of the generating functions  $f_{L_B}(t, p)$  for the first four prime numbers

		$t$				
		$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{4}$
	2		2	$\frac{3}{8}$	$\frac{3}{64}$	$\frac{4}{125}$
	3		8	$\frac{9}{8}$	$\frac{9}{128}$	$\frac{32}{625}$
$p$	5		216	$\frac{27}{2}$	$\frac{27}{512}$	$\frac{864}{15625}$
	7		13512	$\frac{5427}{16}$	$-\frac{243}{4096}$	$-\frac{3936}{78125}$

Since other numerical computations of the generating functions  $f_{L_B}(t, p)$ , for prime numbers greater than 7, contain more digits and fractions with huge numerators and denominators, we give other values of  $f_{L_B}(t, p)$  with some values of  $t$  such as  $t = -1/2$  and  $t = 1/2$ :

The number  $p = 11$  is a member of class of the Lucas primes. Substituting  $p = 11$  into the generating function  $f_{L_B}(t, p)$ , we compute the following series including the generating functions for the numbers of the Lyndon words having length 11 for  $t = -1/2$  and  $t = 1/2$ , respectively

$$f_{L_B}\left(-\frac{1}{2}, 11\right) = \sum_{k=0}^{\infty} L_k(11) \left(-\frac{1}{2}\right)^k = -\frac{33176}{6561},$$

and

$$f_{L_B}\left(\frac{1}{2}, 11\right) = \sum_{k=0}^{\infty} L_k(11) \left(\frac{1}{2}\right)^k = 295024104.$$

The number  $p = 13$  is a Fibonacci prime. Substituting this prime into the generating function  $f_{L_B}(t, p)$ , we compute the value of the following series including the generating functions for the numbers of the Lyndon words having length 13 for  $t = -1/2$  and  $t = 1/2$ , respectively

$$f_{L_B}\left(-\frac{1}{2}, 13\right) = \sum_{k=0}^{\infty} L_k(13) \left(-\frac{1}{2}\right)^k = \frac{1474760}{19683},$$

$$f_{L_B}\left(\frac{1}{2}, 13\right) = \sum_{k=0}^{\infty} L_k(13) \left(\frac{1}{2}\right)^k = 81055130520.$$

If we substitute  $p = 17$ , which is a Fermat prime, into the generating function  $f_{L_B}(t, p)$ , then we compute the following series including the generating functions for the numbers of the Lyndon words having length 17 for  $t = -1/2$  and  $t = 1/2$ , respectively

$$f_{L_B}\left(-\frac{1}{2}, 17\right) = \sum_{k=0}^{\infty} L_k(17) \left(-\frac{1}{2}\right)^k = \frac{11745087800}{531441},$$

$$f_{L_B}\left(\frac{1}{2}, 17\right) = \sum_{k=0}^{\infty} L_k(17) \left(\frac{1}{2}\right)^k = 15337737297545400.$$

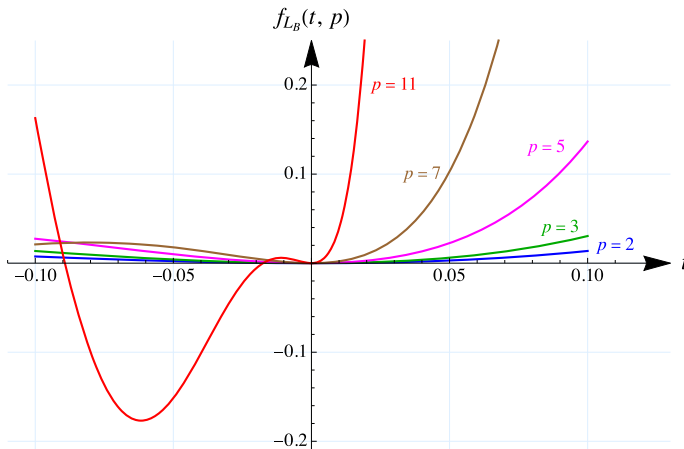


Figure 3. Generating function  $f_{L_B}(t, p)$  for  $p = 2, 3, 5, 7$  and  $11$

### 5. Graphics of the Generating Functions $f_{L_B}(t, p)$

In this section, we draw graphics of the generating functions  $f_{L_B}(t, p)$  and their approximation by rational functions for varying prime numbers.

Firstly, we start with drawing graphics of the generating functions  $f_{L_B}(t, p)$  for varying prime numbers. Therefore, Fig. 3 demonstrates the effects of prime numbers on the shape of the curve for a given range near origin,

$$t \in \left[ -\frac{1}{10}, \frac{1}{10} \right].$$

From Fig. 3, we observe that the shape of graphics of the generating functions  $f_{L_B}(t, p)$  are affected by varying  $p$  values. In this figure, we also observe that when  $p$  values increase, the curves give various different deflection. These curves provide information to analyse characteristics of the generating functions  $f_{L_B}(t, p)$ . These curves may be used to improve some curves for using in geometric design and their applications.

Now, we provide a weighted approximation of the generating functions  $f_{L_B}(t, p)$  by rational functions with small error to decrease algorithmic complexity of the Algorithm 2, which includes combinatorial sums.

Let us assume that  $|t| \leq 1$ . It is well-known that the Apostol–Bernoulli numbers  $\mathcal{B}_n(t)$  are rational functions of variable  $t$ .

Using weighted approximation by these rational functions and the weight function  $t \mapsto w(t) = (1 - t)^2$ , we have

$$\begin{aligned} e_w(t, p) &= |w(t) (f_{L_B}(t, p) - f_A(t, p))| \\ &= w(t) \left| \frac{\mathcal{B}_2(t)}{2^p} - \frac{\mathcal{B}_{p+1}(t)}{p(p+1)} - f_A(t, p) \right| \\ &= \frac{|t|}{p}, \end{aligned} \tag{5.1}$$

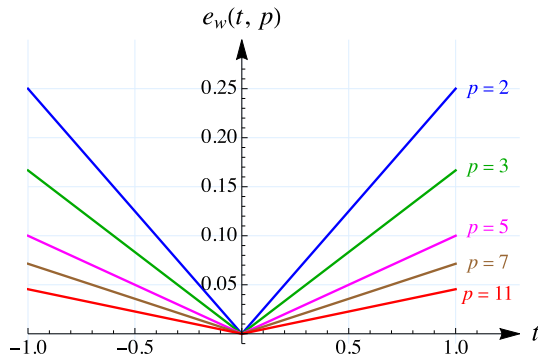


Figure 4. Weighted error  $e_w(t, p)$  for  $p = 2, 3, 5, 7$  and  $11$

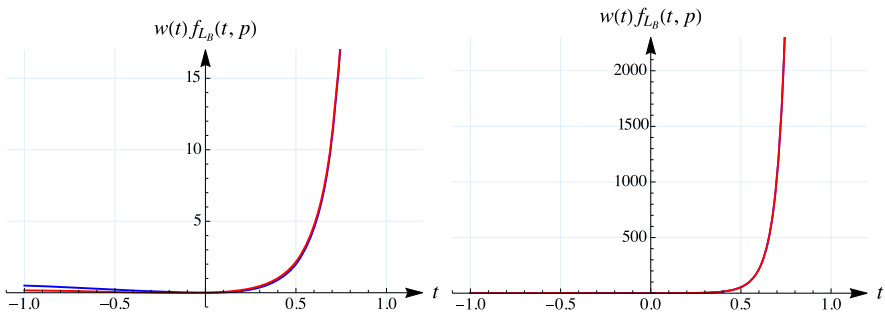


Figure 5. Weighted approximation of the generating function  $f_{L_B}(t, p)$  (blue line) by rational function (red line) for  $p = 3$  (left) and  $p = 5$  (right) on  $[-1, 1]$  (Color figure online)

taking

$$f_A(t, p) = -\frac{\mathcal{B}_{p+1}(t)}{p(p+1)}.$$

Here,  $f_{L_B}(t, p)$  is given by (1.4) and  $\mathcal{B}_2(t) = -2t/(t - 1)^2$ . Thus, we arrive at the following result for prime number  $p$ :

$$\varepsilon_p = \|f_{L_B}(t, p) - f_A(t, p)\|_w = \max_{|t| \leq 1} |w(t) (f_{L_B}(t, p) - f_A(t, p))| = \frac{1}{p}.$$

The errors  $t \mapsto e_w(t, p)$  for prime numbers  $p = 2, 3, 5, 7, 11$  are presented in Fig. 4. In Fig. 5, we present the weighted approximation of the generating function  $f_{L_B}(t, p)$  by the rational function  $f_A(t, p)$  for  $p = 3$  and  $p = 5$ . As we can see we get better approximation when  $p$  increases.

### 6. Exponential Generating Functions for the Numbers of Lyndon Words: Bell Polynomials Approach

In this section, using Bell polynomials (i.e. Exponential functions) approach to the numbers of Lyndon words, we construct the exponential generating functions for the numbers of Lyndon words. Furthermore, we define a new family of special numbers related to these special words, and we investigate their some fundamental properties.

Now, we define exponential generating functions for the numbers  $L_k(n)$  as follows

$$\mathcal{F}(x, n) = \sum_{k=0}^{\infty} L_k(n) \frac{x^k}{k!}. \tag{6.1}$$

By substituting (1.1) into (6.1), we have

$$\mathcal{F}(x, n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \left( \sum_{k=0}^{\infty} k^d \frac{x^k}{k!} \right). \tag{6.2}$$

It is known that the Dobiński's Formula is given by (cf. [35, p. 66])

$$Bl_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k \tag{6.3}$$

where  $Bl_n(x)$  denotes the Bell polynomials (i.e., exponential polynomials) given by (cf. [13], [35, p. 64])

$$Bl_n(x) = \sum_{k=0}^n S(n, k) x^k$$

where  $S(n, k)$  denotes the Stirling numbers of the second kind and the exponential generating function for the Bell polynomials is given by (cf. [13], [35, p. 64])

$$e^{(e^t-1)x} = \sum_{n=0}^{\infty} Bl_n(x) \frac{t^n}{n!}.$$

It is also well-known that the Bell polynomials (i.e., exponential polynomials) satisfy the following two equalities

$$Bl_{n+1}(x) = x \sum_{j=0}^n \binom{n}{j} Bl_j(x) \tag{6.4}$$

and

$$\frac{d}{dx} \{Bl_n(x)\} = \frac{Bl_{n+1}(x)}{x} - Bl_n(x) \tag{6.5}$$

(cf. [4, 9, 13]).

Combining (6.2) and (6.3) we get the following result:

**Theorem 6.1.** *Let  $n$  be a positive integer. Then we have*

$$e^{-x} \mathcal{F}(x, n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) Bl_d(x), \tag{6.6}$$

where the sum is over all positive divisors of  $n$ .

From (6.6), we have

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \sum_{k=0}^{\infty} L_k(n) \frac{x^k}{k!} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) Bl_d(x).$$

Using the Cauchy product in the above equation, we get

$$\sum_{k=0}^{\infty} \left( \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} L_j(n) \right) \frac{x^k}{k!} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) Bl_d(x). \tag{6.7}$$

Motivated by the above computations, we define a new family of special numbers  $\mathcal{I}_n(k)$ , which is related to the numbers  $L_k(n)$ , by the following combinatorial sum

$$\mathcal{I}_n(k) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} L_j(n).$$

The numbers  $\mathcal{I}_n(k)$  are given by the following exponential generating functions

$$\mathcal{F}_{\mathcal{I}}(x, n) = \sum_{k=0}^{\infty} \mathcal{I}_n(k) \frac{x^k}{k!}.$$

Thus, using (6.7), we construct novel exponential generating functions for the numbers  $\mathcal{I}_n(k)$  by the following theorem:

**Theorem 6.2.** *Let  $n$  be a positive integer. Then we have*

$$\mathcal{F}_{\mathcal{I}}(x, n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) Bl_d(x), \tag{6.8}$$

where the sum is over all positive divisors of  $n$ .

Some special cases of the function  $\mathcal{F}_{\mathcal{I}}(x, n)$  is given as follows

$$\begin{aligned} \mathcal{F}_{\mathcal{I}}(x, 1) &= x, \\ \mathcal{F}_{\mathcal{I}}(x, 2) &= \frac{1}{2}x^2, \\ \mathcal{F}_{\mathcal{I}}(x, 3) &= \frac{1}{3}(x^3 + 3x^2), \\ \mathcal{F}_{\mathcal{I}}(x, 4) &= \frac{1}{4}(x^4 + 6x^3 + 6x^2), \end{aligned}$$

etc.

Especially, substituting  $n = p$  (prime number) into (6.8), we arrive at the following result:

**Corollary 6.1.** *Let  $p$  be a prime number. Then*

$$\mathcal{F}_{\mathcal{I}}(x, p) = \frac{Bl_p(x) - Bl_1(x)}{p}.$$

Taking derivative of (6.8) with respect to  $x$ , we get

$$\frac{d}{dx}\{\mathcal{F}_I(x, n)\} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{d}{dx}\{Bl_d(x)\},$$

By combining the above equation with (6.5), we arrive at the following theorem:

**Theorem 6.3.**

$$\frac{d}{dx}\{\mathcal{F}_I(x, n)\} + \mathcal{F}_I(x, n) = \frac{1}{xn} \sum_{d|n} \mu\left(\frac{n}{d}\right) Bl_{d+1}(x). \tag{6.9}$$

By combining (6.9) with (6.4), we also arrive at the following result:

**Theorem 6.4.**

$$\frac{d}{dx}\{\mathcal{F}_I(x, n)\} + \mathcal{F}_I(x, n) = \frac{1}{n} \sum_{d|n} \sum_{k=0}^d \binom{d}{k} \mu\left(\frac{n}{d}\right) Bl_k(x).$$

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**Declarations**

**Conflict of interest** The authors have no conflicts of interest to declare that are relevant to the content of this article.

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**References**

- [1] Apostol, T.M.: On the Lerch zeta function. *Pac. J. Math.* **1**, 161–167 (1951)
- [2] Apostol, T.M.: *Introduction to Analytic Number Theory*. Narosa Publishing, Springer-Verlag, New York, Heidelberg (1976)
- [3] Aygunes, A.A., Simsek, Y.: Unification of multiple Lerch-zeta type functions. *Adv. Stud. Contemp. Math.* **21**(4), 367–373 (2011)
- [4] Bell, E.T.: Exponential polynomials. *Ann. Math. Second Ser.* **35**(2), 258–277 (1934)

- [5] Berndt, B.C.: Ramanujan's Notebooks Part 1: Sums of Powers, Bernoulli Numbers, and the Gamma Function(Chapter 7). Springer, New York (1985)
- [6] Berstel, J., Perrin, D.: The origins of combinatorics on words. *Eur. J. Combin.* **28**, 996–1022 (2007)
- [7] Bona, M.: Introduction to Enumerative Combinatorics. The McGraw-Hill Companies Inc., New York (2007)
- [8] Boyadzhiev, Kh.N.: Apostol-Bernoulli functions, derivative polynomials, and Eulerian polynomials. *Adv. Appl. Discrete Math.* **1**, 109–122 (2008). [arXiv:0710.1124v1](https://arxiv.org/abs/0710.1124v1)
- [9] Boyadzhiev, Kh.N.: Exponential polynomials, stirling numbers, and evaluation of some gamma integrals. *Abstr. Appl. Anal.* **2009**, Article ID 168672, 18 pages (2009). <https://doi.org/10.1155/2009/168672>
- [10] Buchanan, H.L., Knopfmacher, A., Mays, M.E.: On the cyclotomic identity and related product expansions. *Australas. J. Combin.* **8**, 233–245 (1993)
- [11] Cevik, A.S., Das, K.C., Cangul, I.C., Maden, A.D.: Minimality over free monoid presentations. *hacetatepe J. Math. Stat.* **43**(6), 899–913 (2014)
- [12] Charalambides, C.A.: Ennumerative Combinatorics. Chapman&Hall/CRC Press, Company, London, New York (2002)
- [13] Comtet, L.: Advanced Combinatorics. D. Reidel, Dordrecht (1974)
- [14] Cusick, T.W., Stanica, P.: Cryptographic Boolean Functions and Applications. London Academic Press, Elsevier, London (2009)
- [15] Djordjević, G.B., Milovanović, G.V.: Special Classes of Polynomials. University of Nis Faculty of Technology, Leskovac (2014)
- [16] Fredricksen, H., Kessler, I.J.: An algorithm for generating necklaces of beads in two colors. *Discrete Math.* **61**, 181–188 (1986)
- [17] Fredricksen, H., Maiorana, J.: Necklaces of beads in  $k$  colors and  $k$ -ary de Bruijn sequences. *Discrete Math.* **23**, 207–210 (1978)
- [18] Glen, A.: A characterization of fine words over a finite alphabet. *Theor. Comput. Sci.* **391**, 51–60 (2008)
- [19] Good, I.J.: The number of ordering of  $n$  candidates when ties are permitted. *Fibonacci Q.* **13**, 11–18 (1975)
- [20] Hu, S., Kim, M.S.: Two closed forms for the Apostol–Bernoulli polynomials. [arXiv:1509.04190](https://arxiv.org/abs/1509.04190)
- [21] Jang, L.C., Pak, H.K.: Non-Archimedean integration associated with  $q$ -Bernoulli numbers. *Proc. Jangjeon Math. Soc.* **5**(2), 125–129 (2002)
- [22] Kang, S.J., Kim, M.H.: Free Lie algebras, generalized Witt formula, and the denominator identity. *J. Algebra* **183**(2), 560–594 (1996)
- [23] Kim, D.S., Kim, T.: Some new identities of Frobenius–Euler numbers and polynomials. *J. Inequal. Appl.* **307**, 1–10 (2012)
- [24] Kim, T., Rim, S.H., Simsek, Y., Kim, D.: On the analogs of Bernoulli and Euler numbers, related identities and zeta and  $l$ -functions. *J. Korean Math. Soc.* **45**(2), 435–453 (2008)
- [25] Kucukoglu, I., Simsek, Y.: On  $k$ -ary Lyndon words and their generating functions. In: The 14th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM2016). AIP Conference Proceedings, 1863, 2017, 30000-1-30000-4 (2017)

- [26] Lothaire, M.: *Combinatorics on Words*, p. 238. Cambridge University Press, Cambridge (1997)
- [27] Lyndon, R.: On Burnside problem I. *Trans. Am. Math. Soc.* **77**, 202–215 (1954)
- [28] Metropolis, N., Rota, G.C.: Witt vectors and the algebra of necklaces. *Adv. Math.* **50**, 95–125 (1983)
- [29] Milovanović, G.V., Mitrinović, D.S., Rassias, T.M.: *Topics in Polynomials: Extremal Problems, Inequalities*. Zeros. World Scientific Publ. Co., Singapore (1994)
- [30] Ozdemir, G., Simsek, Y., Milovanović, G.V.: Generating functions for special polynomials and numbers including Apostol-type and Humbert-type polynomials. *Mediterr. J. Math.* (2017). <https://doi.org/10.1007/s00009-017-0918-6>
- [31] Ozden, H., Simsek, Y., Srivastava, H.M.: A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials. *Comput. Math. Appl.* **60**, 2779–2787 (2010)
- [32] Ozden, H., Simsek, Y.: Unified presentation of  $p$ -adic  $L$ -functions associated with unification of the special numbers. *Acta Math. Hungar.* **144**(2), 515–529 (2014)
- [33] Petrogradsky, V.M.: Witt’s formula for restricted Lie algebras. *Adv. Appl. Math.* **30**, 219–227 (2003)
- [34] Quaintance, J., Gould, H.W.: *Combinatorial Identities for Stirling Numbers (The Unpublished Notes of H.W. Gould)*. World Scientific Publishing Co., Singapore (2015)
- [35] Roman, S.: *The Umbral Calculus*. Academic Press, New York (1984)
- [36] Ruskey, F., Sawada, J.: An efficient algorithm for generating necklaces with fixed density. *SIAM J. Comput.* **29**(2), 671–684 (1999)
- [37] Simsek, Y.:  $q$ -Analogue of the twisted  $l$ -series and  $q$ -twisted Euler numbers. *J. Number Theory* **100**(2), 267–278 (2005)
- [38] Simsek, Y.: Twisted  $p$ -adic  $(h, q)$ - $L$ -functions. *Comput. Math. Appl.* **59**, 2097–2110 (2010)
- [39] Simsek, Y.: Generating functions for generalized stirling type numbers, array type polynomials, Eulerian type polynomials and their applications. *Fixed Point Theory A* **2013**(87), 1–28 (2013)
- [40] Simsek, Y.: Identities associated with generalized Stirling type numbers and Eulerian type polynomials. *Math. Comput. Appl.* **18**(3), 251–263 (2013)
- [41] Simsek, Y.: Apostol type Dahee numbers and polynomials. *Adv. Stud. Contemp. Math.* **26**(3), 1–12 (2016)
- [42] Simsek, Y.: On generating functions for the special polynomials. *Filomat* **31**(1), 9–16 (2017)
- [43] Simsek, Y., Kim, T., Park, D.W., Ro, Y.S., Jang, L.C., Rim, S.H.: An explicit formula for the multiple Frobenius–Euler numbers and polynomials. *JP J. Algebra Number Theory Appl.* **4**(3), 519–529 (2004)
- [44] Simsek, Y., Yurekli, O., Kurt, V.: On interpolation functions of the twisted generalized Frobenius–Euler numbers. *Adv. Stud. Contemp. Math.* **15**(2), 187–194 (2007)
- [45] Simsek, Y., Srivastava, H.M.: A family of  $p$ -adic twisted interpolation functions associated with the modified Bernoulli numbers. *Appl. Math. Comput.* **216**, 2976–2987 (2010)

- [46] Srivastava, H.M.: Some formulas for the Bernoulli and Euler polynomials at rational arguments. *Math. Proc. Camb. Philos. Soc.* **129**, 77–84 (2000)
- [47] Srivastava, H.M., Manocha, H.L.: *A Treatise on Generating Functions*. Ellis Horwood Limited Publisher, Chichester (1984)
- [48] Srivastava, H.M., Choi, J.: *Series Associated with the Zeta and Related Functions*. Kluwer Academic Publishers, Dordrecht (2001)
- [49] Srivastava, H.M., Tomovski, Z., Leskovski, D.: Some families of Mathieu type series and Hurwitz–Lerch Zeta functions and associated probability distributions. *Appl. Comput. Math.* **14**(3), 349–380 (2015). (**Special Issue**)
- [50] Srivastava, H.M., Ozden, H., Cangul, I.N., Simsek, Y.: A unified presentation of certain meromorphic functions related to the families of the partial zeta type functions and the  $L$ -functions. *Appl. Math. Comput.* **219**, 3903–3913 (2012)
- [51] Srivastava, H.M., Choi, J.: *Zeta and  $q$ -Zeta Functions and Associated Series and Integrals*. Elsevier Science Publishers, Amsterdam (2012)
- [52] Witt, E.: *Treue Darstellung Liescher Ringe*. *J. Reine Angew. Math.* **177**, 152–160 (1937)
- [53] [https://en.wikipedia.org/wiki/Category:Classes\\_of\\_prime\\_numbers](https://en.wikipedia.org/wiki/Category:Classes_of_prime_numbers)

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