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On an inverse problem for a nonlinear third order in time partial differential equation

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ABSTRACT

In this article, first we convert an inverse problem of determining the unknown timewise terms of nonlinear third order in time partial differential equation (PDE) from knowledge of two boundary measurements to the auxiliary system of integral equations. Then, existence and uniqueness of the solution of this system is proved by means of the contraction principle on a small time interval. Also uniqueness of the solution of the inverse problem is given. However, since the governing equation is yet ill-posed (very slight errors in the temperature input may cause relatively significant errors in the output potential and source terms), we need to regularize the solution. Therefore, to get a stable solution, a regularized cost function is to be minimized for retrieval of the unknown terms. The third order in time PDE problem is discretized using the FDM and reshaped as non-linear least-squares optimization of the Tikhonov regularization function. This is numerically solved by means of the MATLAB subroutine *lsqnonlin* tool. Both analytical and perturbed data are inverted. Numerical outcomes for two benchmark test examples are reported and discussed. The proposed numerical approach has also been discussed.

1. Introduction

Third order in time PDEs arise in many scientific fields such as non-linear acoustics, medical ultrasound, viscoelasticity and thermoelasticity, see [1–8] to mention only a few. The inverse problems for a third order in time PDEs connected with recovery of the coefficients are scarce and attracted by many authors in recent decade. Using Carleman estimates determination of the space dependent damping coefficient for the linearized third order in time PDE (referred as Jordan–Moore–Gibson–Thompson equation) is studied in [9] with known one boundary measurement and for Moore–Gibson–Thompson equation is investigated in [10] from partial knowledge of some trace of the solution at the boundary. Inverse problems of recovering the time dependent lowest coefficient under simultaneous over-determination conditions for the specific third order in time PDEs are investigated in [11–13]. Finally in [14], the authors proved the existence and uniqueness of the solution of the inverse problem for general non-linear third order in time PDE by considering the critical parameter is positive.

Recently, the authors in [15] discussed the existence of unique solution of a class of third order pseudoparabolic inverse boundary value problems subject to no-classical conditions of first kind. Khompysh [16] investigated the reconstruction

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of unknown coefficient in pseudo-parabolic inverse problem with the integral over determination condition and studied the uniqueness and existence of solution by means of method of successive approximations. Ramazanova et al. [17] theoretically studied the fourth-order inverse problem with non-local integral condition. Serikbaev and Tokmagambetov [18] determined the space dependent forcing term for a class of pseudoparabolic inverse problems. Huntul et al. [19] studied the reconstruction of unknown time and space-dependent coefficients in third-order pseudo-parabolic inverse problems subject to nonlocal integral conditions. Further, Huntul et al. [20,21] considered a fourth order pseudoparabolic inverse problem with some additional information and identified the unknown time dependent potential coefficient. Recently, Mehraliyev and Shayeva [22] investigated an inverse problem theoretically for the third order pseudoparabolic problem and proved the uniqueness of the solution. The existence and uniqueness of the solution of the inverse problem for the third order pseudoparabolic equation with integral over-determination condition is studied in [12]. However, the work on inverse problems for third order in time PDE is scarce.

In this article, we study the third order in time PDE problem to reconstruct the time wise potential and source terms theoretically, i.e. existence and uniqueness of the auxiliary integral problem, and numerically, for the first time, in the prescribed domain using the initial, Neumann boundary conditions, and the temperature data as an over-determination conditions. The pre-eminent goal of the current work is to undertake the theory and numerical aspect of this problem.

This article is presented as follows. The nonlinear third order in time PDE has been mathematically developed in Section 2. In Section 3, the inverse problem is converted to an auxiliary system of integral equations. Then, uniqueness of the solution of the inverse problem and existence and uniqueness of the solution of this system are proved by means of the contraction principle on a small time interval. Section 4 briefly explains the scheme for solving the forward problem by means of FDM technique. The description of numerical procedure to solve the minimization of the nonlinear Tikhonov regularization functional has been given in Section 5. The numerical outcomes for some examples on the topic are discussed in Section 6. The conclusions remarks are revealed in Section 7.

2. Formulation of the nonlinear third order problem

In this section, as a mathematical model, we consider an inverse problem of reconstructing the time-dependent terms $a(\tau)$ and $b(\tau)$ in the non-linear third order in time partial differential equation

$$z_{\tau\tau\tau} + \beta z_{\tau\tau} - c^2 z_{\xi\xi} - \gamma z_{\tau\xi\xi} = a(\tau)f(z) + b(\tau)g(\xi,\tau), \quad (\xi,\tau) \in Q_T, \tag{1}$$

where $z = z(\xi,\tau)$, $Q_T = \{(\xi,\tau) : 0 \le \xi \le 1, 0 \le \tau \le T\}$, $\beta, c^2, \gamma > 0$ are given constants, with initial conditions

$$z(\xi,0) = \phi_0(\xi), \quad z_\tau(\xi,0) = \phi_1(\xi), \quad z_{\tau\tau}(\xi,0) = \phi_2(\xi), \quad \xi \in [0,1], \tag{2}$$

the boundary conditions

$$z_\xi(0,\tau) = z_\xi(1,\tau) = 0, \quad \tau \in [0,T], \tag{3}$$

and the over-determination temperature conditions

$$z(0,\tau) = h_1(\tau), \quad \tau \in [0,T], \tag{4}$$

$$z(1,\tau) = h_2(\tau), \quad \tau \in [0,T], \tag{5}$$

where $\phi_0(\xi)$, $\phi_1(\xi)$, $\phi_2(\xi)$, $g(\xi,\tau)$, $f(z)$, $h_1(\tau)$ and $h_2(\tau)$ are known functions while $a(\tau)$, $b(\tau)$ and $z(\xi,\tau)$ are desired functions. This model is the normalized form of the model which has been firstly introduced by Moore and Walter in [23] and given in original form by Thompson in [24]. The third order in time Eqs. (1) has a critical parameter $\Upsilon \equiv \beta - \frac{c^2}{\gamma}$. This parameter plays an important role in asymptotic behaviour, energy estimates and regularity of solutions. Indeed, all studies so far requires the positivity assumption $\Upsilon > 0$. This is the common case considered in non-linear acoustics, where $\Upsilon\gamma$ is equal to the Lighthill's diffusivity of sound, which is always positive, [25]. However, when $\Upsilon = 0$ the energy is conservative and no decay of the energy occurs, despite the presence of diffusivity and damping, [26–28]. The aim of this paper is determining the time-dependent coefficient $a(\tau)$ multiplying the non-linear term and time dependent force coefficient $b(\tau)$ together with the $z(\xi,\tau)$ from Eq. (1), initial conditions (2), homogeneous Neumann boundary conditions (3) and additional conditions (4) and (5) under the assumption $\Upsilon = 0$ (energy is conservative).

3. Uniqueness of the solution of the inverse problem

In this section, first we will examine the auxiliary spectral problem of the inverse problem (1)–(5) and give two functional spaces which are Banach spaces. Then, we will set and prove the existence and uniqueness theorem for a solution to the auxiliary integral problem, from which the uniqueness of the solution to the inverse problem will follow.

The auxiliary spectral problem of the inverse problem (1)–(5) is

$$\begin{cases} X''(\xi) + \lambda X(\xi) = 0, \quad 0 \le \xi \le 1, \\ X'(0) = X'(1) = 0. \end{cases} \tag{6}$$

The spectral problem (6) has eigenvalues $\lambda_n = (n\pi)^2$ and corresponding eigenfunctions $X_n(\xi) = \sqrt{2}\cos(\sqrt{\lambda_n}\xi)$, $n = 0, 1, 2, \ldots$. It is easy to verify that the system of eigenfunctions $X_n(\xi)$ are biorthonormal on $[0, 1]$ and forms a Riesz basis in $L_2[0, 1]$.

Definition 1. Let the triplet of functions $\{z(\xi,\tau), a(\tau), b(\tau)\}$ be from the class $C^{2,3}(Q_T) \times C[0, T]$ and satisfies Eq. (1) and conditions (2)–(5). Then the pair triplet $\{z(\xi,\tau), a(\tau), b(\tau)\}$ is called the classical solution of the inverse problem (1)–(5).

Consider the following useful Banach spaces:

I:

$$B_T = \Bigg\{ z(\xi,\tau) = \sum_{n=0}^{\infty} z_n(\tau)X_n(\xi) : \ z_n(\tau) \in C[0, T],$$

$$J_T(z) = \|z_0(\tau)\|_{C[0,T]} + \left(\sum_{n=1}^{\infty} (\lambda_n^{3/2} \|z_n(\tau)\|_{C[0,T]})^2 \right)^{1/2} < +\infty \Bigg\},$$

with the norm $\|z(\xi,\tau)\|_{B_T} \equiv J_T(u)$ which is related with the Fourier coefficients of the function $z(\xi,\tau)$ by the eigenfunctions $X_n(\xi)$, $n = 0, 1, 2, \ldots$.

II: $E_T = B_T \times C[0, T]$ of the vector function $w(\xi,\tau) = \{z(\xi,\tau), a(\tau), b(\tau)\}$ with the norm

$$\|w(\xi,\tau)\|_{E_T} = \|z(\xi,\tau)\|_{B_T} + \|a(\tau)\|_{C[0,T]} + \|b(\tau)\|_{C[0,T]}.$$

These spaces are suitable to investigate the solution of the inverse problem (1)–(5).

For arbitrary $a(\tau), b(\tau) \in C[0, T]$, to construct the formal solution of the inverse problem (1)–(5), we will use the Fourier (Eigenfunction expansion) method. In accordance with this, let us consider

$$z(\xi,\tau) = \sum_{n=0}^{\infty} z_n(\tau)X_n(\xi) \tag{7}$$

is a solution of the inverse problem (1)–(5), where $X_n(\xi) = \sqrt{2}\cos(\sqrt{\lambda_n}\xi)$, $n = 0, 1, 2, \ldots$ are the eigenfunctions and $\lambda_n = (n\pi)^2$, $n = 0, 1, 2, \ldots$ are the eigenvalues of the corresponding spectral problem.

Since $z(\xi,\tau)$ is the formal solution of the inverse problem (1)–(5), we get the following Cauchy problems with respect to $z_n(\tau)$ from Eq. (1) and initial conditions (2);

$$\begin{cases} z_n'''(\tau) + \beta z_n''(\tau) + \gamma\lambda_n z_n'(\tau) + c^2\lambda_n z_n(\tau) = F_n(\tau; a, b, z), \\ z_n(0) = \phi_{0n}, \ z_n'(0) = \phi_{1n}, \ z_n''(0) = \phi_{2n}, \ n = 0, 1, 2, \ldots. \end{cases} \tag{8}$$

Here

$$F_n(\tau; a, b, u) = a(\tau)f_n(\tau) + b(\tau)g_n(\tau),$$

$$z_n(\tau) = \int_0^1 z(\xi,\tau)X_n(\xi)d\xi,$$

$$f_n(\tau) = \int_0^1 f(z(\xi,\tau))X_n(\xi)d\xi,$$

$$g_n(\tau) = \int_0^1 g(\xi,\tau)X_n(\xi)d\xi,$$

and

$$\phi_{in} = \sqrt{2}\int_0^1 \phi_i(\xi)X_n(\xi)d\xi, \, i = 0, 1, 2, n = 0, 1, 2, \ldots.$$

These Cauchy problems have the cubic characteristic polynomial

$$P_3(k) = k^3 + \beta k^2 + \gamma\lambda_n k + c^2\lambda_n.$$

Then the roots of the characteristic polynomial are

$$k_1 = -\frac{c^2}{\gamma} < 0 \text{ and } k_{2,3} = \pm i\sqrt{\gamma\lambda_n}.$$

Solving (8) by using the these roots of the characteristic polynomial, we obtain

$$z_0(\tau) = \phi_{00} + \tau\phi_{10} + \left(\frac{e^{-\beta\tau} + \beta\tau - 1}{\beta^2}\right)\phi_{20} + \int_0^\tau F_0(s; a, b, z)\left(\frac{\tau - s}{\beta} + \frac{e^{-\beta(\tau-s)} - 1}{\beta^2}\right) ds, \tag{9}$$

$$z_n(\tau) = A_{0n}(\tau)\phi_{0n} + A_{1n}(\tau)\phi_{1n} + A_{2n}(\tau)\phi_{2n} + \int_0^\tau A_{3n}(\tau - s)F_n(s; a, b, z)ds, \tag{10}$$

where

$$A_{0n}(\tau) = \frac{\gamma^3\lambda_n e^{-\frac{c^2}{\gamma}\tau} + c^4\cos(\sqrt{\gamma\lambda_n}\tau) + c^2\sqrt{\gamma^3\lambda_n}\sin(\sqrt{\gamma\lambda_n}\tau)}{c^4 + \gamma^3\lambda_n},$$

$$A_{1n}(\tau) = \frac{1}{\sqrt{\gamma\lambda_n}}\sin(\sqrt{\gamma\lambda_n}\tau),$$

$$A_{2n}(\tau) = \frac{\gamma^2\sqrt{\lambda_n}e^{-\frac{c^2}{\gamma}\tau} - \gamma^2\sqrt{\lambda_n}\cos(\sqrt{\gamma\lambda_n}\tau) + c^2\sqrt{\gamma}\sin(\sqrt{\gamma\lambda_n}\tau)}{\sqrt{\lambda_n}\left(c^4 + \gamma^3\lambda_n\right)},$$

$$A_{3n}(\tau) = \frac{\gamma^2\left(\sqrt{\gamma\lambda_n}e^{-\frac{c^2}{\gamma}\tau} + \frac{c^2}{\gamma}\sin(\sqrt{\gamma\lambda_n}\tau) - \sqrt{\gamma\lambda_n}\cos(\sqrt{\gamma\lambda_n}\tau)\right)}{\sqrt{\gamma\lambda_n}\left(c^4 + \gamma^3\lambda_n\right)}.$$

Substitute the expressions (9) and (10) into (7) to determine $z(\xi, \tau)$. Then we get

$$\begin{aligned} z(\xi, \tau) =& \Bigg(\phi_{00} + \tau\phi_{10} + \left(\frac{e^{-\beta\tau} + \beta\tau - 1}{\beta^2}\right)\phi_{20} \\ &+ \int_0^\tau F_0(s; a, b, z)\left(\frac{\tau - s}{\beta} + \frac{e^{-\beta(\tau-s)} - 1}{\beta^2}\right) ds\Bigg) X_0(\xi) \\ &+ \sum_{n=1}^{\infty}\Bigg(A_{0n}(\tau)\phi_{0n} + A_{1n}(\tau)\phi_{1n} + A_{2n}(\tau)\phi_{2n} \\ &+ \int_0^\tau A_{3n}(\tau - s)F_n(s; a, b, z)ds\Bigg)X_n(\xi). \end{aligned} \tag{11}$$

Let us derive the equations of $a(\tau)$ and $b(\tau)$. Considering the over-determination conditions (4) and (5) into Eq. (1), we obtain the system of equations for $a(\tau)$ and $b(\tau)$ as follows:

$$a(\tau)f(h_1(\tau)) + b(\tau)g(0, \tau) = h_1'''(\tau) + \beta h_1''(\tau) - c^2 z_{\xi\xi}(0, \tau) - \gamma z_{\tau\xi\xi}(0, \tau), \tag{12}$$

$$a(\tau)f(h_2(\tau)) + b(\tau)g(1, \tau) = h_2'''(\tau) + \beta h_2''(\tau) - c^2 z_{\xi\xi}(1, \tau) - \gamma z_{\tau\xi\xi}(1, \tau). \tag{13}$$

Since

$$z(\xi, \tau) = \sum_{n=0}^{\infty} z_n(\tau)X_n(\xi), \quad z_{\xi\xi}(\xi, \tau) = -\sum_{n=1}^{\infty}\lambda_n z_n(\tau)X_n(\xi),$$

$$z_{\tau\xi\xi}(\xi, \tau) = -\sum_{n=1}^{\infty}\lambda_n z_n'(\tau)X_n(\xi),$$

$$z_{\xi\xi}(0, \tau) = -\sum_{n=1}^{\infty}\lambda_n z_n(\tau)X_n(0) = -\sum_{n=1}^{\infty}\lambda_n z_n(\tau),$$

$$z_{\tau\xi\xi}(0, \tau) = -\sum_{n=1}^{\infty}\lambda_n z_n'(\tau)X_n(0) = -\sum_{n=1}^{\infty}\lambda_n z_n'(\tau),$$

$$z_{\xi\xi}(1, \tau) = -\sum_{n=1}^{\infty}\lambda_n z_n(\tau)X_n(1) = -\sum_{n=1}^{\infty}(-1)^n\lambda_n z_n(\tau),$$

and

$$z_{\tau\xi\xi}(1,\tau) = -\sum_{n=1}^{\infty} \lambda_n z_n'(\tau) X_n(1) = -\sum_{n=1}^{\infty} (-1)^n \lambda_n z_n'(\tau).$$

Thus, we get

$$a(\tau)f(h_1(\tau)) + b(\tau)g(0,\tau) = h_1'''(\tau) + \beta h_1''(\tau) + \sum_{n=1}^{\infty} \lambda_n \left[\gamma z_n'(\tau) + c^2 z_n(\tau)\right], \tag{14}$$

$$a(\tau)f(h_2(\tau)) + b(\tau)g(1,\tau) = h_2'''(\tau) + \beta h_2''(\tau) + \sum_{n=1}^{\infty} (-1)^n \lambda_n \left[\gamma z_n'(\tau) + c^2 z_n(\tau)\right], \tag{15}$$

where

$$\gamma z_n'(\tau) + c^2 z_n(\tau) = \widehat{A}_{0n}(\tau)\phi_{0n} + \widehat{A}_{1n}(\tau)\phi_{1n} + \widehat{A}_{2n}(\tau)\phi_{2n} + \int_0^{\tau} \widehat{A}_{3n}(\tau - s) F_n(s; a, b, z) ds,$$

with

$$\widehat{A}_{0n}(\tau) = \frac{\left(c^2\gamma^3\lambda_n + c^6\right)\cos(\sqrt{\gamma\lambda_n}\tau)}{c^4 + \gamma^3\lambda_n},$$

$$\widehat{A}_{1n}(\tau) = \gamma\cos(\sqrt{\gamma\lambda_n}\tau) + \frac{c^2}{\sqrt{\gamma\lambda_n}}\sin(\sqrt{\gamma\lambda_n}\tau),$$

$$\widehat{A}_{2n}(\tau) = \frac{\left(\gamma^{7/2}\lambda_n + c^4\sqrt{\gamma}\right)\sin(\sqrt{\gamma\lambda_n}\tau)}{\left(c^4 + \gamma^3\lambda_n\right)\sqrt{\lambda_n}},$$

$$\widehat{A}_{3n}(\tau) = \int_0^{\tau} \frac{\left(\gamma c^4 + \gamma^4\lambda_n\right)\sin(\sqrt{\gamma\lambda_n}(\tau - s))}{\left(c^4 + \gamma^3\lambda_n\right)\sqrt{\gamma\lambda_n}}.$$

Under the assumption $\Delta(\tau) = f(h_1(\tau))g(1,\tau) - f(h_2(\tau))g(0,\tau) \neq 0$, $\forall \tau \in [0,T]$, finally we can derive the equations of $a(\tau)$ and $b(\tau)$ as

$$a(\tau) = \frac{1}{\Delta(\tau)}\Bigg[g(1,\tau)h_1'''(\tau) - g(0,\tau)h_2'''(\tau) + \beta\left(g(1,\tau)h_1''(\tau) - g(0,\tau)h_2''(\tau)\right) + \sum_{n=1}^{\infty} \lambda_n \left(g(1,\tau) + (-1)^{n+1}g(0,\tau)\right)\left(\gamma z_n'(\tau) + c^2 z_n(\tau)\right)\Bigg], \tag{16}$$

$$b(\tau) = \frac{1}{\Delta(\tau)}\Bigg[f(h_1(\tau))h_2'''(\tau) - f(h_2(\tau))h_1'''(\tau) + \beta\left(f(h_1(\tau))h_2''(\tau) - f(h_2(\tau))h_1''(\tau)\right) + \sum_{n=1}^{\infty} \lambda_n \left((-1)^n f(h_1(\tau)) - f(h_2(\tau))\right)\left(\gamma z_n'(\tau) + c^2 z_n(\tau)\right)\Bigg]. \tag{17}$$

We obtained the system of Volterra integral Eqs. (11), (16) and (17) with respect to $z(\xi,\tau)$, $a(\tau)$ and $b(\tau)$. Analogously, we can prove that if $\{z(\xi,\tau), a(\tau), b(\tau)\}$ is a solution of the inverse problem (1)–(5), then

$$z_n(\tau) = \int_0^1 z(\xi,\tau)X_n(\xi)d\xi, \quad n = 0, 1, 2, \ldots$$

satisfy the system of differential Eqs. (8). For the proof of this assertion please see [29]. From this assertion we can conclude that proving the uniqueness of the solution of the inverse problem (1)–(5), it suffices to prove the uniqueness of the solution of the system (11), (16) and (17).

Theorem 1. *Let the assumptions*

(A_1) $\phi_0(\xi) \in C^2[0,1]$, $\phi_0'''(\xi) \in L_2[0,1]$, $\phi_0'(0) = \phi_0'(1) = 0$;

(A_2) $\phi_1(\xi) \in C^2[0,1]$, $\phi_1'''(\xi) \in L_2[0,1]$, $\phi_1'(0) = \phi_1'(1) = 0$;

(A_3) $\phi_2(\xi) \in C^2[0,1]$, $\phi_2'''(\xi) \in L_2[0,1]$, $\phi_2'(0) = \phi_2'(1) = 0$;

(A_4) $h_1(\tau) \in C^3[0,T]$, $h_1(0) = \phi_0(0)$, $h_1'(0) = \phi_1(0)$, $h_1''(0) = \phi_2(0)$;

(A_5) $h_2(\tau) \in C^3[0,T]$, $h_2(0) = \phi_0(1)$, $h_2'(0) = \phi_1(1)$, $h_2''(0) = \phi_2(1)$;

(A_6) $g(\xi,\tau) \in C(Q_T)$, $g_\xi, g_{\xi\xi} \in C[0,1]$, $\forall \tau \in [0,T]$, $g_\xi(0,\tau) = g_\xi(1,\tau) = 0$;

$$\textbf{(A}_7\textbf{)} \quad \begin{cases} f(z) \in C^2(\mathbb{R}), \\ \left|f(z^1) - f(z^2)\right| \le \varepsilon_1 \left|z^1 - z^2\right|, \\ \left|f'(z^1) - f'(z^2)\right| \le \varepsilon_2 \left|z^1 - z^2\right|, \end{cases} \quad \varepsilon_i > 0, \ i = 1, 2,$$

be satisfied, $\beta = \frac{c^2}{\gamma}$ *and* $\Delta(\tau) = f(h_1(\tau))g(1,\tau) - f(h_2(\tau))g(0,\tau) \neq 0$, $\forall \tau \in [0, T]$. *Then, the system* (11), (16) and (17) *has a unique solution for small* T.

Proof. To prove the existence of a unique solution of the system (11), (16) and (17) we need to rewrite this system into operator form and to show that this operator a contraction operator. To this end let us denote $w(\xi,\tau) = [z(\xi,\tau), a(\tau), b(\tau)]^T$ is a 3×1 vector function and rewrite the system of Eqs. (11), (16) and (17) in the following operator equation

$$w = \Psi(w), \tag{18}$$

where $\Psi(w) \equiv [\psi_1, \psi_2, \psi_3]^T$ and ψ_1, ψ_2 and ψ_3 are equal to the right hand sides of (11), (16) and (17), respectively. i.e.

$$\begin{aligned} \psi_1(w) = &\left(\phi_{00} + \tau\phi_{10} + \left(\frac{e^{-\beta\tau} + \beta\tau - 1}{\beta^2}\right)\phi_{20} \right.\\ &\left. + \int_0^\tau F_0(s; a, b, z)\left(\frac{\tau - s}{\beta} + \frac{e^{-\beta(\tau-s)} - 1}{\beta^2}\right) ds\right) X_0(\xi) \\ &+ \sum_{n=1}^{\infty}\left(A_{0n}(\tau)\phi_{0n} + A_{1n}(\tau)\phi_{1n} + A_{2n}(\tau)\phi_{2n}\right. \\ &\left. + \int_0^\tau A_{3n}(\tau - s)F_n(s; a, b, z)ds\right) X_n(\xi), \end{aligned} \tag{19}$$

$$\begin{aligned} \psi_2(w) = &\frac{1}{\Delta(\tau)}\left[g(1,\tau)h_1'''(\tau) - g(0,\tau)h_2'''(\tau) + \beta\left(g(1,\tau)h_1''(\tau) - g(0,\tau)h_2''(\tau)\right)\right. \\ &\left. + \sum_{n=1}^{\infty}\lambda_n\left(g(1,\tau) + (-1)^{n+1}g(0,\tau)\right)\left(\gamma z_n'(\tau) + c^2 z_n(\tau)\right)\right], \end{aligned} \tag{20}$$

and

$$\begin{aligned} \psi_3(w) = &\frac{1}{\Delta(\tau)}\left[f(h_1(\tau))h_2'''(\tau) - f(h_2(\tau))h_1'''(\tau) + \beta\left(f(h_1(\tau))h_2''(\tau) - f(h_2(\tau))h_1''(\tau)\right)\right. \\ &\left. + \sum_{n=1}^{\infty}\lambda_n\left((-1)^n f(h_1(\tau)) - f(h_2(\tau))\right)\left(\gamma z_n'(\tau) + c^2 z_n(\tau)\right)\right]. \end{aligned} \tag{21}$$

Using integration by parts under the assumptions $(A_1) - (A_7)$, we have

$$\phi_{0n} = \frac{1}{\lambda_n^{3/2}}\alpha_{0n}, \ \phi_{1n} = \frac{1}{\lambda_n^{3/2}}\alpha_{1n}, \ \phi_{2n} = \frac{1}{\lambda_n^{3/2}}\alpha_{2n}, \ g_n(\tau) = \frac{1}{\lambda_n}\omega_n(\tau), \ f_n(t) = \frac{1}{\lambda_n}\upsilon_n(\tau),$$

where

$$\alpha_{in} = \sqrt{2}\int_0^1 \phi_i'''(\xi)\sin(\sqrt{\lambda_n}\xi)d\xi, \ i = 0, 1, 2,$$

$$\omega_n(\tau) = -\sqrt{2}\int_0^1 g_{\xi\xi}(\xi,\tau)\cos(\sqrt{\lambda_n}\xi)d\xi,$$

and

$$\upsilon_n(\tau) = -\sqrt{2}\int_0^1 \left(f''(z)(z_\xi)^2 + f'(z)z_{\xi\xi}\right)\cos(\sqrt{\lambda_n}\xi)d\xi.$$

Since $\sqrt{2}\sin(\sqrt{\lambda_n}\xi)$ (or $\sqrt{2}\cos(\sqrt{\lambda_n}\xi)$) forms a biorthogonal system of functions on $[0, 1]$, by using Bessel's inequality we get

$$\sum_{n=1}^{\infty}|\alpha_{in}|^2 \le \left\|\phi_i'''\right\|^2_{L_2[0,1]}, \ i = 0, 1, 2,$$

$$\sum_{n=1}^{\infty}|\upsilon_n(\tau)|^2 \le \left\|f''(z_\xi(\cdot,\tau))^2 + f'z_{\xi\xi}(\cdot,\tau)\right\|^2_{L_2[0,1]}, \quad \sum_{n=1}^{\infty}|\omega_n(\tau)|^2 \le \left\|g_{\xi\xi}(\cdot,\tau)\right\|^2_{L_2[0,1]}. \tag{22}$$

By virtue of the definition of the space B_T, using the Cauchy–Schwarz inequality it is easy to obtain that

$$\left|z_\xi\right| = \left|\sqrt{2}\sum_{n=1}^{\infty}\sqrt{\lambda_n}z_n(\tau)\sin(\sqrt{\lambda_n}\xi)\right| \le \sqrt{2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left(\lambda_n^{3/2}\left\|z_n(\tau)\right\|_{C[0,T]}\right) \le \rho_1\left\|z\right\|_{B_T},$$

$$\left|z_{\xi\xi}\right| = \left|\sqrt{2}\sum_{n=1}^{\infty}\lambda_n z_n(\tau)\cos(\sqrt{\lambda_n}\xi)\right| \le \sqrt{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}}\left(\lambda_n^{3/2}\left\|z_n(\tau)\right\|_{C[0,T]}\right) \le \rho_2\left\|z\right\|_{B_T}, \tag{23}$$

where ρ_i, $i = 1, 2$ are real constants.

Also we can obtain the following estimates for the coefficients arising in the operator Eqs. (19)–(21)

$$|A_{in}(\tau)| \le d_i, \quad |A_{3n}(\tau)| \le \frac{d_3}{\sqrt{\lambda_n}}, \quad \left|\widehat{A}_{in}(\tau)\right| \le D_i, \quad \left|\widehat{A}_{3n}(\tau)\right| \le \frac{D_3}{\sqrt{\lambda_n}}, i = 0, 1, 2, \tag{24}$$

by considering $A_{in}(\tau)$ and $\widehat{A}_{in}(\tau)$, $i = 0, 1, 2, 3$ are convergent sequences as $\lambda_n \to +\infty$ $(n \to +\infty)$. Here d_i and D_i, $i = 0, 1, 2, 3$ are positive constants.

After giving these assumptions and estimates, we can show in two steps that Ψ is a contraction operator.

(I) First let us demonstrate that Ψ maps E_T onto itself continuously. In other words, for arbitrary $w = [z(\xi,\tau), a(\tau), b(\tau)]^T$ with $z(\xi,\tau) \in B_T$, $a(\tau), b(\tau) \in C[0,T]$ we need to show $\psi_1(w) \in B_T$ and $\psi_2(w), \psi_3(w), \in C[0,T]$.

Let us start with showing that $\psi_1(w) \in B_T$, i.e. we need to verify

$$J_T(\psi_1) = \left\|\psi_{1,0}(t)\right\|_{C[0,T]} + \left(\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\left\|\psi_{1,n}(\tau)\right\|_{C[0,T]}\right)^2\right)^{1/2} < +\infty, \tag{25}$$

where

$$\psi_{1,0}(\tau) = \phi_{00} + \tau\phi_{10} + \left(\frac{e^{-\beta\tau} + \beta\tau - 1}{\beta^2}\right)\phi_{20} + \int_0^\tau F_0(s; a, b, z)\left(\frac{\tau - s}{\beta} + \frac{e^{-\beta(\tau-s)} - 1}{\beta^2}\right)ds,$$

$$\psi_{1,n}(\tau) = A_{0n}(\tau)\phi_{0n} + A_{1n}(\tau)\phi_{1n} + A_{2n}(\tau)\phi_{2n} + \int_0^\tau A_{3n}(\tau - s)F_n(s; a, b, z)ds.$$

After some manipulations under the assumptions $(A_1) - (A_7)$, using the estimates given above, we obtain

$$\left\|\psi_{1,0}(\tau)\right\|_{C[0,T]} \le |\phi_{00}| + T|\phi_{10}| + \frac{\beta T + 2}{\beta^2}|\phi_{20}| + \frac{\beta T^2 + 2T}{\beta^2}\left(\max_{0\le\tau\le T}|a(\tau)|\max_{0\le\tau\le T}|f_0(t)| + \max_{0\le\tau\le T}|b(\tau)|\max_{0\le\tau\le T}|g_0(t)|\right),$$

$$\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\left\|\psi_{1,n}(\tau)\right\|_{C[0,T]}\right)^2 \le 5d_0^2\sum_{n=1}^{\infty}|\alpha_{0n}|^2 + 5d_1^2\sum_{n=1}^{\infty}|\alpha_{1n}|^2 + 5d_2^2\sum_{n=1}^{\infty}|\alpha_{2n}|^2 + 5d_3^2T^2\left(\left(\max_{0\le\tau\le T}|a(\tau)|\right)^2\sum_{n=1}^{\infty}\left(\max_{0\le\tau\le T}|\upsilon_n(t)|\right)^2 + \left(\max_{0\le\tau\le T}|b(\tau)|\right)^2\sum_{n=1}^{\infty}\left(\max_{0\le t\le T}|\omega_n(t)|\right)^2\right).$$

From the Bessel inequality (considering the estimates (22)) we can conclude that the right hand side of $J_T(\psi_1)$ is convergent. Thus $J_T(\psi_1) < +\infty$ and $\psi_1(w)$ belongs to the space B_T.

Now let us verify $\psi_2(w), \psi_3(w) \in C[0,T]$. From the Eqs. (20) and (21), we have

$$|\psi_2(w)| \le \frac{1}{\min\limits_{0\le\tau\le T}|\Delta(\tau)|}\left[\left|g(1,\tau)h_1'''(\tau)\right| + \left|g(0,\tau)h_2'''(\tau)\right| + \beta\left(\left|g(1,\tau)h_1''(\tau)\right| + \left|g(0,\tau)h_2''(\tau)\right|\right) + \sum_{n=1}^{\infty}\lambda_n\left(|g(1,\tau)| + |g(0,\tau)|\right)\left(\left|\gamma z_n'(\tau) + c^2 z_n(\tau)\right|\right)\right],$$

and

$$|\psi_3(w)| = \frac{1}{\min\limits_{0\le\tau\le T}|\Delta(\tau)|}\left[\left|f(h_1(\tau))h_2'''(\tau)\right| + \left|f(h_2(\tau))h_1'''(\tau)\right| + \beta\left(\left|f(h_1(\tau))h_2''(\tau)\right| + \left|f(h_2(\tau))h_1''(\tau)\right|\right) + \sum_{n=1}^{\infty}\lambda_n\left(|f(h_1(\tau))| + |f(h_2(\tau))|\right)\left(\left|\gamma z_n'(\tau) + c^2 z_n(\tau)\right|\right)\right].$$

Using the Cauchy–Schwarz inequality and the estimates are given in (22) and (24) we obtain

$$\begin{aligned}
\|\psi_2(w)\|_{C[0,T]} \le{}& \frac{1}{\min\limits_{0\le\tau\le T}|\Delta(\tau)|}\Bigg[\max_{0\le\tau\le T}|g(1,\tau)|\max_{0\le\tau\le T}\left|h_1'''(\tau)\right| \\
&+\max_{0\le\tau\le T}|g(0,\tau)|\max_{0\le\tau\le T}\left|h_2'''(\tau)\right|+\beta\left(\max_{0\le\tau\le T}|g(1,\tau)|\max_{0\le\tau\le T}\left|h_1''(\tau)\right|\right. \\
&\left.+\max_{0\le\tau\le T}|g(0,\tau)|\max_{0\le\tau\le T}\left|h_2''(\tau)\right|\right)+\left(\max_{0\le\tau\le T}|g(1,\tau)|\right. \\
&\left.+\max_{0\le\tau\le T}|g(0,\tau)|\right)\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\right)^{1/2}\left\{\sum_{i=0}^{2}D_i\left(\sum_{n=1}^{\infty}|\alpha_{in}|^2\right)^{1/2}\right. \\
&+D_3T\left\{\max_{0\le\tau\le T}|a(\tau)|\left(\sum_{n=1}^{\infty}\left(\max_{0\le\tau\le T}|\upsilon_n(t)|\right)^2\right)^{1/2}\right. \\
&\left.\left.+\max_{0\le\tau\le T}|b(\tau)|\left(\sum_{n=1}^{\infty}\left(\max_{0\le t\le T}|\omega_n(t)|\right)^2\right)^{1/2}\right\}\right\}\Bigg],
\end{aligned}\tag{26}$$

and

$$\begin{aligned}
\|\psi_3(w)\|_{C[0,T]} \le{}& \frac{1}{\min\limits_{0\le\tau\le T}|\Delta(\tau)|}\Bigg[\max_{0\le\tau\le T}|f(h_1(\tau))|\max_{0\le\tau\le T}\left|h_2'''(\tau)\right| \\
&+\max_{0\le\tau\le T}|f(h_2(\tau))|\max_{0\le\tau\le T}\left|h_1'''(\tau)\right|+\beta\left(\max_{0\le\tau\le T}|f(h_1(\tau))|\max_{0\le\tau\le T}\left|h_2''(\tau)\right|\right. \\
&\left.+\max_{0\le\tau\le T}|f(h_2(\tau))|\max_{0\le\tau\le T}\left|h_1''(\tau)\right|\right)+\left(\max_{0\le\tau\le T}|f(h_1(\tau))|\right. \\
&\left.+\max_{0\le\tau\le T}|f(h_2(\tau))|\right)\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\right)^{1/2}\left\{\sum_{i=0}^{2}D_i\left(\sum_{n=1}^{\infty}|\alpha_{in}|^2\right)^{1/2}\right. \\
&+D_3T\left\{\max_{0\le\tau\le T}|a(\tau)|\left(\sum_{n=1}^{\infty}\left(\max_{0\le\tau\le T}|\upsilon_n(t)|\right)^2\right)^{1/2}\right. \\
&\left.\left.+\max_{0\le\tau\le T}|b(\tau)|\left(\sum_{n=1}^{\infty}\left(\max_{0\le t\le T}|\omega_n(t)|\right)^2\right)^{1/2}\right\}\right\}\Bigg].
\end{aligned}\tag{27}$$

Considering the estimates (22) and $\sum_{n=0}^{\infty}\frac{1}{\lambda_n}$ is convergent, the majorizing series (26) and (27) are convergent. According to Weierstrass M-test, $\psi_2(w)$, $\psi_3(w)$ are continuous and belongs to the space $C[0,T]$. Therefore, we show that Ψ maps E_T onto itself continuously.

(II) Now let us show that Ψ is contraction mapping operator. Assume that let w_1 and w_2 be any two elements of E_T. We know that

$$\begin{aligned}
\|\Psi(w_1)-\Psi(w_2)\|_{E_T} &= \|\psi_1(w_1)-\psi_1(w_2)\|_{B_T} \\
&\quad+\|\psi_2(w_1)-\psi_2(w_2)\|_{C[0,T]}+\|\psi_3(w_1)-\psi_3(w_2)\|_{C[0,T]}.
\end{aligned}$$

Here $w_i=\left[z^{(i)}(\xi,\tau),a^{(i)}(\tau),b^{(i)}(\tau)\right]^T$, $i=1,2$. Let us consider the following differences

$$\begin{aligned}
&\psi_1(w_1)-\psi_1(w_2) \\
&\quad=\int_0^{\tau}\left[F_0(s;a^{(1)},b^{(1)},z^{(1)})-F_0(s;a^{(2)},b^{(2)},z^{(2)})\right]\left(\frac{\tau-s}{\beta}+\frac{e^{-\beta(\tau-s)}-1}{\beta^2}\right)dsX_0(x) \\
&\qquad+\sum_{n=1}^{\infty}\int_0^{\tau}A_{3n}(\tau-s)\left[F_n(s;a^{(1)},b^{(1)},z^{(1)})-F_n(s;a^{(2)},b^{(2)},z^{(2)})\right]dsX_n(x),
\end{aligned}$$

$$\psi_2(w_1)-\psi_2(w_2)=\frac{1}{\Delta(\tau)}\sum_{n=1}^{\infty}\lambda_n\left(g(1,\tau)\right.$$

$$+(-1)^{n+1}g(0,\tau)\Big)\int_0^\tau \widehat{A}_{3n}(\tau-s)\Big[F_n(s;a^{(1)},b^{(1)},z^{(1)}) - F_n(s;a^{(2)},b^{(2)},z^{(2)})\Big]ds,$$

and

$$\psi_3(w_1)-\psi_3(w_2)=\frac{1}{\Delta(\tau)}\sum_{n=1}^{\infty}\lambda_n\Big((-1)^n f(h_1(\tau)) - f(h_2(\tau))\Big)\int_0^\tau \widehat{A}_{3n}(\tau-s)\Big[F_n(s;a^{(1)},b^{(1)},z^{(1)})-F_n(s;a^{(2)},b^{(2)},z^{(2)})\Big]ds.$$

After some manipulations in last equations under the assumptions (A_1)–(A_7) and using the estimates (22)–(24), we obtain

$$\begin{aligned}\|\Psi(w_1)-\Psi(w_2)\|_{E_T} &= \|\psi_1(w_1)-\psi_1(w_2)\|_{B_T}+\|\psi_2(w_1)-\psi_2(w_2)\|_{C[0,T]}\\ &+\|\psi_3(w_1)-\psi_3(w_2)\|_{C[0,T]}\le K(T)C(a^{(1)},z^{(1)},z^{(2)})\Big[\left\|z^{(1)}-z^{(2)}\right\|_{B_T}\\ &+\left\|a^{(1)}-a^{(2)}\right\|_{C[0,T]}+\left\|b^{(1)}-b^{(2)}\right\|_{C[0,T]}\Big]=K(T)C(a^{(1)},z^{(1)},z^{(2)})\,\|w_1-w_2\|_{E_T}\,,\end{aligned}\tag{28}$$

where

$$K(T)=T\left[1+\frac{2}{\min\limits_{0\le\tau\le T}|\Delta(\tau)|}\right],$$

and $C(a^{(1)},z^{(1)},z^{(2)})$ is the constant depends on the norms $\left\|a^{(1)}\right\|_{C[0,T]}$ and $\left\|z^{(i)}\right\|_{B_T}$, $(i=1,2)$ and the coefficients ρ_j, ε_j, $(j=1,2)$, d_k and D_k, $(k=0,1,2,3\ )$. For sufficiently small T such as $K(T)$ tends to zero, i.e. $0<K(T)C(a^{(1)},z^{(1)},z^{(2)})<1$ for sufficiently small T. This implies that the operator Ψ is contraction mapping operator. Thus from the first and second steps the operator Ψ is contraction mapping and maps E_T onto itself continuously. Then according to Banach fixed point theorem there exists unique solution of (18). □

From this theorem we can conclude that the uniqueness of the solution of the inverse problem (1)–(5) as follows:

Corollary 1. *Let the assumptions of Theorem 1 and*

$$0<K(T)C(a^{(1)},z^{(1)},z^{(2)})<1$$

are satisfied. Then if the solution of the inverse problem (1)–(5) exists than it is unique for sufficiently small T.

4. Discretization of the nonlinear third order problem

We consider the direct problem (1)–(3). When $a(\tau)$, $b(\tau)$, $f(z)$, γ, β, c^2 are known and $z(\xi,\tau)$ is to be found. First, we divide the domain Q_T into M and N intervals of equal lengths $\Delta\xi=\frac{1}{M}$ and $\Delta\tau=\frac{T}{N}$. We denote $z(\xi_i,\tau_j)=z_{i,j}$, $g(\xi_i,\tau_j)=g_{i,j}$, $a(\tau_j)=a^j$, $b(\tau_j)=b^j$, where $\xi_i=i\Delta\xi$, $\tau_j=j\Delta\tau$, for $i=0,1,\dots,M$ and $j=0,1,\dots,N$. The nonlinear term $f(z)$ is tackled by evaluating $f(z)$ at a time level of the previous steps known solutions. Finally, the FDM [30] discretizes Eq. (1) as

$$\begin{aligned}&\frac{z_{i,j+2}-2z_{i,j+1}+2z_{i,j-1}-z_{i,j-2}}{2(\Delta\tau)^3}+\beta\left(\frac{z_{i,j+1}-2z_{i,j}+z_{i,j-1}}{(\Delta\tau)^2}\right)\\ &-c^2\left(\frac{z_{i+1,j}-2z_{i,j}+z_{i-1,j}}{(\Delta\xi)^2}\right)-\gamma\left(\frac{z_{i+1,j+1}-2z_{i,j+1}+z_{i-1,j+1}}{2\Delta\tau(\Delta\xi)^2}\right.\\ &\left.-\frac{z_{i+1,j-1}-2z_{i,j-1}+z_{i-1,j-1}}{2\Delta\tau(\Delta\xi)^2}\right)=a^jf(z_{i,j})+b^jg_{i,j},\\ &i=0,1,\dots,M,\quad j=0,1,\dots,N.\end{aligned}\tag{29}$$

Simplifying above equation, we get

$$\begin{aligned}z_{i,j+2}&=Az_{i+1,j+1}+Bz_{i,j+1}+Az_{i-1,j+1}+Cz_{i+1,j}+Dz_{i,j}+Cz_{i-1,j}-Az_{i+1,j-1}\\ &+Ez_{i,j-1}-Az_{i-1,j-1}-z_{i,j-2}+R_{i,j},\quad i=1,2,\dots,M-1,\quad j=2,3,\dots,N,\end{aligned}\tag{30}$$

where

$$A=\frac{\gamma(\Delta\tau)^2}{(\Delta\xi)^2},\quad B=2-2\beta\Delta\tau-\frac{2\gamma(\Delta\tau)^2}{(\Delta\xi)^2},\quad C=\frac{2c^2(\Delta\tau)^3}{(\Delta\xi)^2}$$

$$D = 4\beta\Delta\tau - \frac{4c^2(\Delta\tau)^3}{(\Delta\xi)^2}, \quad E = -2 - 2\beta\Delta\tau + \frac{2\gamma(\Delta\tau)^2}{(\Delta\xi)^2}$$
$$R_{i,j} = 2(\Delta\tau)^3\left(a^j f(z_{i,j}) + b^j g_{i,j}\right), \quad i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N.$$

Now, we discretize the initial and boundary conditions as

$$z(\xi, 0) = \phi_0(\xi) \implies z_{i,0} = \phi_0(\xi_i), \quad i = 0, 1, \dots, M, \tag{31}$$

$$z_\tau(\xi, 0) = \phi_1(\xi) \implies z_{i,-1} = z_{i,1} - 2\Delta\tau\phi_1(\xi_i), \quad i = 0, 1, \dots, M, \tag{32}$$

$$\begin{aligned} z_{\tau\tau}(\xi, 0) = \phi_2(\xi) \implies z_{i,-2} &= -z_{i,2} + 32z_{i,1} - 30z_{i,0} \\ &-32\Delta\tau\phi_1(\xi_i) - 12(\Delta\tau)^2\phi_2(\xi_i), \quad i = 0, 1, \dots, M, \end{aligned} \tag{33}$$

$$z_\xi(0, \tau) = 0 \implies z_{-1,j} = z_{1,j}, \quad j = 0, 1, \dots, N, \tag{34}$$

and

$$z_\xi(1, \tau) = 0 \implies z_{M+1,j} = z_{M-1,j}, \quad j = 0, 1, \dots, N. \tag{35}$$

For $i = 0$ and $j = 0$, using (32), (33) and (34) in (30), we get

$$\begin{aligned} z_{0,2} = &-\frac{1}{2}(B + E - 32)z_{0,1} - \frac{1}{2}(D + 30)z_{0,0} - Cz_{1,0} - (16 - E)\Delta\tau\phi_1(\xi_0) \\ &-2A\Delta\tau\phi_1(\xi_1) - 6(\Delta\tau)^2\phi_2(\xi_0) + \frac{1}{2}R_{0,0}. \end{aligned} \tag{36}$$

For $i = 1, 2, \dots, M - 1$ and $j = 0$, using (32) and (33) in (30), we get

$$\begin{aligned} z_{i,2} = &-\frac{1}{2}(B + E - 32)z_{i,1} - \frac{1}{2}Cz_{i-1,0} - \frac{1}{2}(D + 30)z_{i,0} - \frac{1}{2}Cz_{i+1,0} \\ &-A\Delta\tau\phi_1(\xi_{i-1}) - (16 - E)\Delta\tau\phi_1(\xi_i) - A\Delta\tau\phi_1(\xi_{i+1}) - 6(\Delta\tau)^2\phi_2(\xi_i) + R_{i,0}. \end{aligned} \tag{37}$$

For $i = M$ and $j = 0$, using (32), (33) and (35) in (30), we get

$$\begin{aligned} z_{M,2} = &-\frac{1}{2}(B + E - 32)z_{M,1} - Cz_{M-1,0} - \frac{1}{2}(D + 30)z_{M,0} \\ &-2A\Delta\tau\phi_1(\xi_{M-1}) - (16 - E)\Delta\tau\phi_1(\xi_M) - 6(\Delta\tau)^2\phi_2(\xi_M) + \frac{1}{2}R_{M,0}. \end{aligned} \tag{38}$$

For $i = 0$ and $j = 1$, using (32) and (34) in (30), we get

$$z_{0,3} = -Bz_{0,2} - 2Az_{1,2} - (D - 1)z_{0,1} - 2Cz_{1,1} - Ez_{0,0} + 2Az_{1,0} - 2\Delta\tau\phi_1(\xi_0) + R_{0,1}. \tag{39}$$

For $i = 1, 2, \dots, M - 1$ and $j = 1$, using (32) in (30), we get

$$\begin{aligned} z_{i,3} = &-Az_{i-1,2} - Bz_{i,2} - Az_{i+1,2} - Cz_{i-1,1} - (D - 1)z_{i,1} - Cz_{i+1,1} \\ &+ Az_{i-1,0} - Ez_{i,0} + Az_{i+1,0} - 2\Delta\tau\phi_1(\xi_i) + R_{i,1}. \end{aligned} \tag{40}$$

For $i = M$ and $j = 1$, using (32) and (35) in (30), we get

$$\begin{aligned} z_{M,3} = &-2Az_{M-1,2} - Bz_{M,2} - 2Cz_{M-1,1} - (D - 1)z_{M,1} \\ &+ 2Az_{M-1,0} - Ez_{M,0} - 2\Delta\tau\phi_1(\xi_M) + R_{M,1}. \end{aligned} \tag{41}$$

For $i = 0$, using (34) in (30), we get

$$\begin{aligned} z_{0,j+2} = &-2Az_{1,j+1} - Bz_{0,j+1} - 2Cz_{1,j} - Dz_{0,j} + 2Az_{1,j-1} - Ez_{0,j-1} \\ &+ z_{0,j-2} + R_{0,j}, \quad j = 2, 3, \dots, N. \end{aligned} \tag{42}$$

Finally, for $i = M$, using (35) in (30), we get

$$\begin{aligned} z_{M,j+2} = &-2Az_{M-1,j+1} - Bz_{M,j+1} - 2Cz_{M-1,j} - Dz_{M,j} + +2Az_{M-1,j-1} \\ &- Ez_{M,j-1} + z_{M,j-2} + R_{M,j}, \quad j = 2, 3, \dots, N. \end{aligned} \tag{43}$$

Table 1
The RMSE between the analytical (46) and approximate values for $h_1(\tau)$ and $h_2(\tau)$, with $M = N \in \{10, 20, 40\}$.

$M = N$	10	20	40
RMSE(h_1)	4.9368E−4	1.2097E−4	2.9928E−5
RMSE(h_2)	4.9368E−4	1.2097E−4	2.9928E−5

Table 2
The exact (46) and (47) and approximate solutions for $h_1(\tau)$ and $h_2(\tau)$ for the direct problem with $\Delta\kappa = \Delta\tau \in \left\{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}\right\}$.

τ	0.1	0.2	0.3	...	0.8	0.9	$\Delta\kappa = \Delta\tau$
$h_1(\tau)$	0.905000	0.819000	0.741199	...	0.449914	0.407194	$\frac{1}{10}$
	0.904874	0.818798	0.740910	...	0.449477	0.406721	$\frac{1}{20}$
	0.904846	0.818747	0.740841	...	0.449366	0.406607	$\frac{1}{40}$
	0.904837	**0.818730**	**0.740818**	...	**0.449328**	**0.406569**	exact
$h_2(\tau)$	−0.905000	−0.819000	−0.741199	...	−0.449914	−0.407194	$\frac{1}{10}$
	−0.904874	−0.818798	−0.740910	...	−0.449477	−0.406721	$\frac{1}{20}$
	−0.904846	−0.818747	−0.740841	...	−0.449366	−0.406607	$\frac{1}{40}$
	−**0.904837**	−0.818730	−**0.740818**		−**0.449328**	−**0.406569**	exact

The theoretical error analysis of the FDM scheme in Eq. (29) using the Taylor expansion estimate is deferred to future work.

Example. Consider the following example of the forward problem (1)–(3) with $T = 1$ and

$$\begin{aligned}
&\phi_0(\xi) = z(\xi, 0) = \cos(\pi\xi),\ \ \phi_1(\xi) = z_\tau(\xi, 0) = -\cos(\pi\xi),\ \ \phi_2(\xi) = z_{\tau\tau}(\xi, 0) = \cos(\pi\xi),\\
&z_\xi(0, \tau) = z_\xi(1, \tau) = 0,\ \ \gamma = 1,\ \ \beta = 1,\ \ c^2 = 1,\ \ f(z) = e^z,\ \ a(\tau) = 1 + 2\tau,\\
&b(\tau) = 1 + \tau,\ \ g(\xi, \tau) = -\frac{e^{e^{-\tau}\cos(\pi\xi)}(1 + 2\tau)}{1 + \tau}.
\end{aligned} \tag{44}$$

The analytical exact solution is

$$z(\xi, \tau) = e^{-\tau}\cos(\pi\xi), \quad (\xi, \tau) \in \Omega_T, \tag{45}$$

and the target outputs (4) and (5) are

$$h_1(\tau) = e^{-\tau}, \tag{46}$$

$$h_2(\tau) = -e^{-\tau}. \tag{47}$$

The three dimensional visuals of approximate and the exact solutions are shown in Fig. 1 using different values of $\Delta\xi$ and $\Delta\tau$ and we can see that the computational results converge to the analytical exact solution as the mesh sizes are decreased. Table 2 depicts the numerical additional measurements in ((4) and (5)) in comparison of the exact (46) and (47) obtained by using the FDM as described in Section 3. The RMS errors for $M = N = 10, 20, 40$ are also listed in Table 1.

5. Solution of the nonlinear inverse problem

We want to get stable and accurate determinations of $a(\tau)$, $b(\tau)$ and $z(\kappa, \tau)$ that satisfy (1)–(4). The proposed problem is numerically solved by minimizing the following regularized cost function

$$\mathbb{F}(a, b) = \|z(0, \tau) - h_1(\tau)\|^2 + \|z(1, \tau) - h_2(\tau)\|^2 + \delta\left(\|a(\tau)\|^2 + \|b(\tau)\|^2\right), \tag{48}$$

where z satisfies the forward problem (1)–(3) with given $a(\tau)$, $b(\tau)$, and δ is the non-negative penalty parameter. The discretized form of above function is given by

$$\mathbb{F}(\underline{a}, \underline{b}) = \sum_{j=1}^{N}\left[z(0, \tau_j) - h_1(\tau_j)\right]^2 + \sum_{j=1}^{N}\left[z(1, \tau_j) - h_2(\tau_j)\right]^2 + \delta\left(\sum_{j=1}^{N}(a^j)^2 + \sum_{j=1}^{N}(b^j)^2\right). \tag{49}$$

The cost function (49) is minimizing by means of the MATLAB subroutine *lsqnonlin* tool [31].

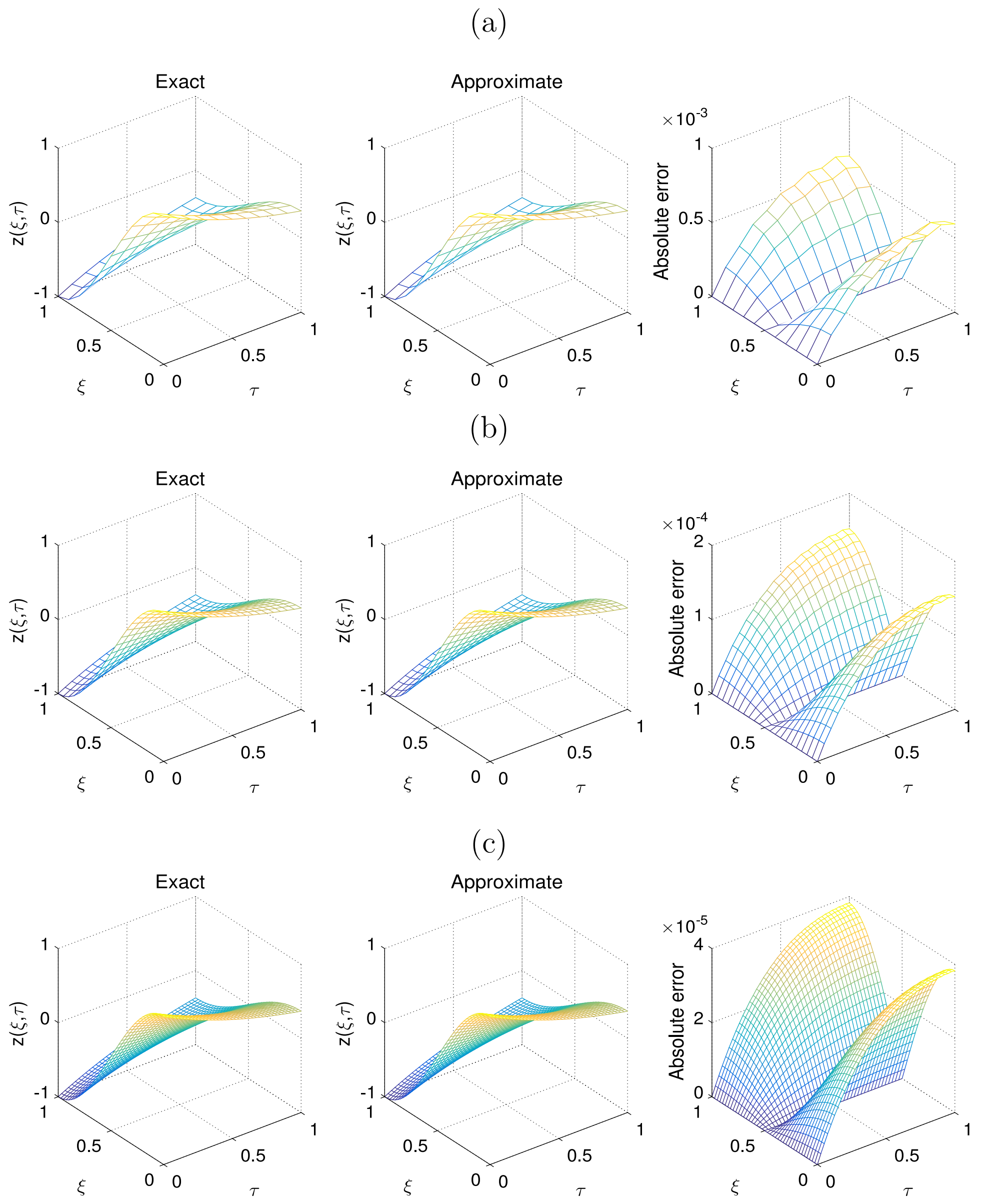

Fig. 1. The analytical (45) solutions, approximate curves and the absolute computational error for the forward problem using $\Delta\xi = \Delta\tau = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$.

6. Numerical results and discussion

A couple of experiment examples have been discussed in this section to demonstrate the stability and accuracy of the FDM together with the Tikhonov regularization process. To validate the efficiency of the numerical solution, the root mean square error (RMSE) has been calculated as:

$$\mathrm{RMSE}(a) = \left[\frac{T}{N}\sum_{j=1}^{N}\left(a^{numerical}(\tau_j) - a^{exact}(\tau_j)\right)^2\right]^{1/2}, \tag{50}$$

$$\mathrm{RMSE}(b) = \left[\frac{T}{N}\sum_{j=1}^{N}\left(b^{numerical}(\tau_j) - b^{exact}(\tau_j)\right)^2\right]^{1/2}. \tag{51}$$

For the sake of simplicity, we set $T = 1$. The upper and lower bounds for $a(\tau)$ and $b(\tau)$ are supposed to be 10^2 and -10^2, respectively.

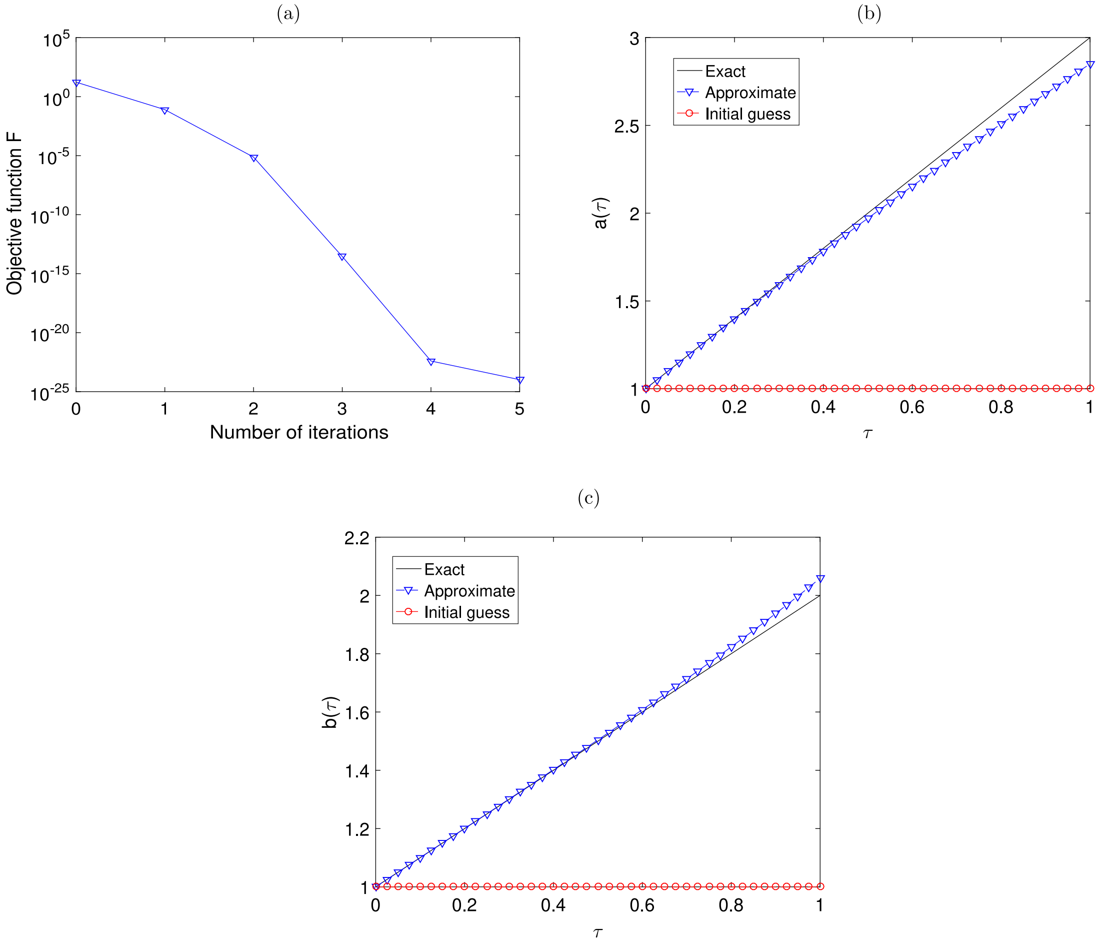

Fig. 2. (a) The unregularized form of objective function $\mathbb{F}$(49) versus no. of iterations, and the approximate and analytical exact curves (55) for: (b) the potential $a(\tau)$ and (c) the source $b(\tau)$, in absence of noise and regularization, for Example 1.

The problem (1)–(4) has been solved with both perturbed and exact data. The perturbed data is handled as

$$h_1^{\epsilon_1}(\tau_j) = h_1(\tau_j) + \epsilon 1_j, \quad j = 1, 2, \ldots, N, \tag{52}$$

$$h_2^{\epsilon_2}(\tau_j) = h_2(\tau_j) + \epsilon 2_j, \quad j = 1, 2, \ldots, N, \tag{53}$$

where $\epsilon 1_j$ and $\epsilon 2_j$ denote the random variables obtained from a Gaussian normal distribution having zero mean and the following standard deviations:

$$\sigma_1 = \max_{0 \le \tau \le T} |h_1(\tau)| \times p, \quad \sigma_2 = \max_{0 \le \tau \le T} |h_2(\tau)| \times p, \tag{54}$$

where p denotes the percentage of noise. For the perturbed data (52), (53), $h_1(\tau_j)$ and $h_2(\tau_j)$ are replaced by $h_1^{\epsilon_1}(\tau_j)$, $h_2^{\epsilon_2}(\tau_j)$ in (49).

6.1. Example 1

As a first test problem, we take (1)–(4) with input (44)–(47), and the unknown smooth and linear time-dependent terms $a(\tau)$ and $b(\tau)$

$$a(\tau) = 1 + 2\tau, \quad b(\tau) = 1 + \tau, \quad \tau \in [0, 1]. \tag{55}$$

From given data it can be observed that the assumptions of Theorem 1 are fulfilled, so, the inverse problem possesses a unique solution. The initial approximation for $\underline{a}$ and $\underline{b}$ are taken as

$$a^0(\tau_j) = a(0) = 1, \quad b^0(\tau_j) = b(0) = 1, \quad j = 1, 2, \ldots, N. \tag{56}$$

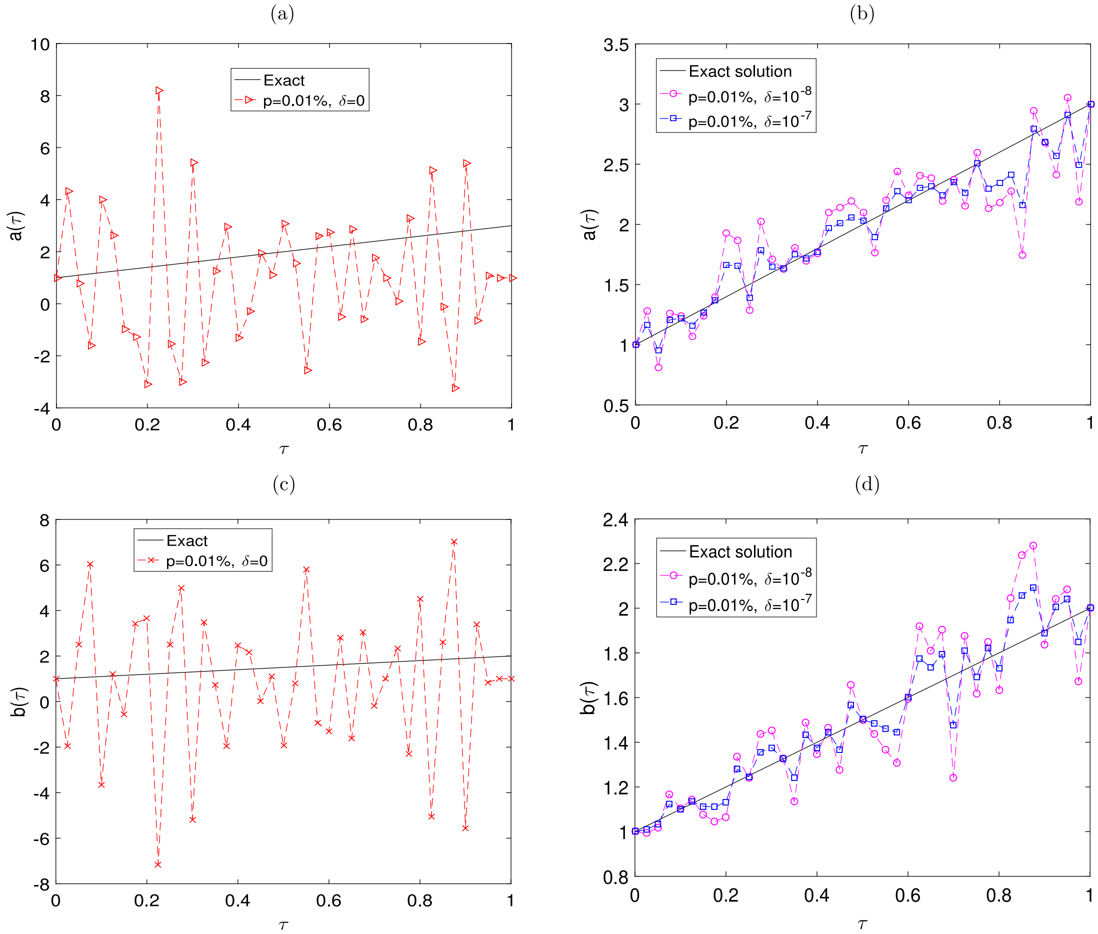

Fig. 3. The approximate and analytical (55) solutions of the potential $a(\tau)$, and the heat source $b(\tau)$, for $p = 0.01\%$, with $\delta \in \{0, 10^{-8}, 10^{-7}\}$, for Example 1.

We fix $\Delta\xi = \Delta\tau = 0.025$. In Fig. 2(a), the unregularized function (49), i.e. $\delta = 0$ has been plotted versus number of iterations in absence of noise in measurement data. It can be noted that after six iterations, there is a quick decline low value of $O(10^{-24})$. The exact (55) and approximate solutions to the function $a(\tau)$ and $b(\tau)$ are portrayed in Figs. 2(b) and 2(c). It is observed that the numerical outcomes are accurate with $\text{RMSE}(a) = 0.0662$ and $\text{RMSE}(b) = 0.0206$.

Next, we associate 0.01%, 0.1% noise with the simulated data (4) and (5), as in Eq. (54). It is significant to note that the inverse problem is not well posed therefore, we anticipate that the cost function needs to be regularized for the sake of stability and accuracy in results. Although not presented, it is illustrated that the regularized cost function $\mathbb{F}$ versus no. of iterations monotonically decreasing convergence is observed. Figs. 3 and 4 show visuals of the reconstructed terms $a(\tau)$ and $b(\tau)$. From Figs. 3(a), 3(c) and 4(a), 4(c) it is clear that, as expected, for $\delta = 0$ we obtain inaccurate and unstable solutions with $\text{RMSE}(a) = 2.8624$ and $\text{RMSE}(b) = 3.3265$ for $p = 0.01\%$, and $\text{RMSE}(a) = 27.8127$ and $\text{RMSE}(b) = 32.09511$ for $p = 0.1\%$, respectively, as the problem is noise sensitive and ill-posed. Hence, regularization process is crucial for stable solutions. For this, the regularization parameter δ is chosen to be 10^{-8}, 10^{-7} for $p = 0.01\%$ noise (see Figs. 3(b) and 3(d) obtaining $\text{RMSE}(a) \in \{0.3045, 0.1689\}$ and $\text{RMSE}(b) \in \{0.1794, 0.0912\}$), and $\delta \in \{10^{-6}, 10^{-5}\}$ for $p = 0.1\%$ noise (see Figs. 4(b) and 4(d) obtaining $\text{RMSE}(a) \in \{0.3591, 0.3045\}$ and $\text{RMSE}(b) \in \{0.2151, 0.1794\}$, which provide stable and comparatively accurate approximations for the functions $a(\tau)$ and $b(\tau)$. The numerically obtained results are stable and accurate. The main difficulty in regularization when we solve the nonlinear inverse problem of third-order in time is how to choose an appropriate regularization parameter δ which compromises between accuracy and stability. However, one can use techniques such as the L-curve method [32] or Morozov's discrepancy principle [33] to find such a parameter, but in our work we have used trial and error. As mentioned in [34], the regularization parameter δ is selected based on experience by first choosing a small value and gradually increasing it until any numerical oscillations in the unknown coefficient disappear.

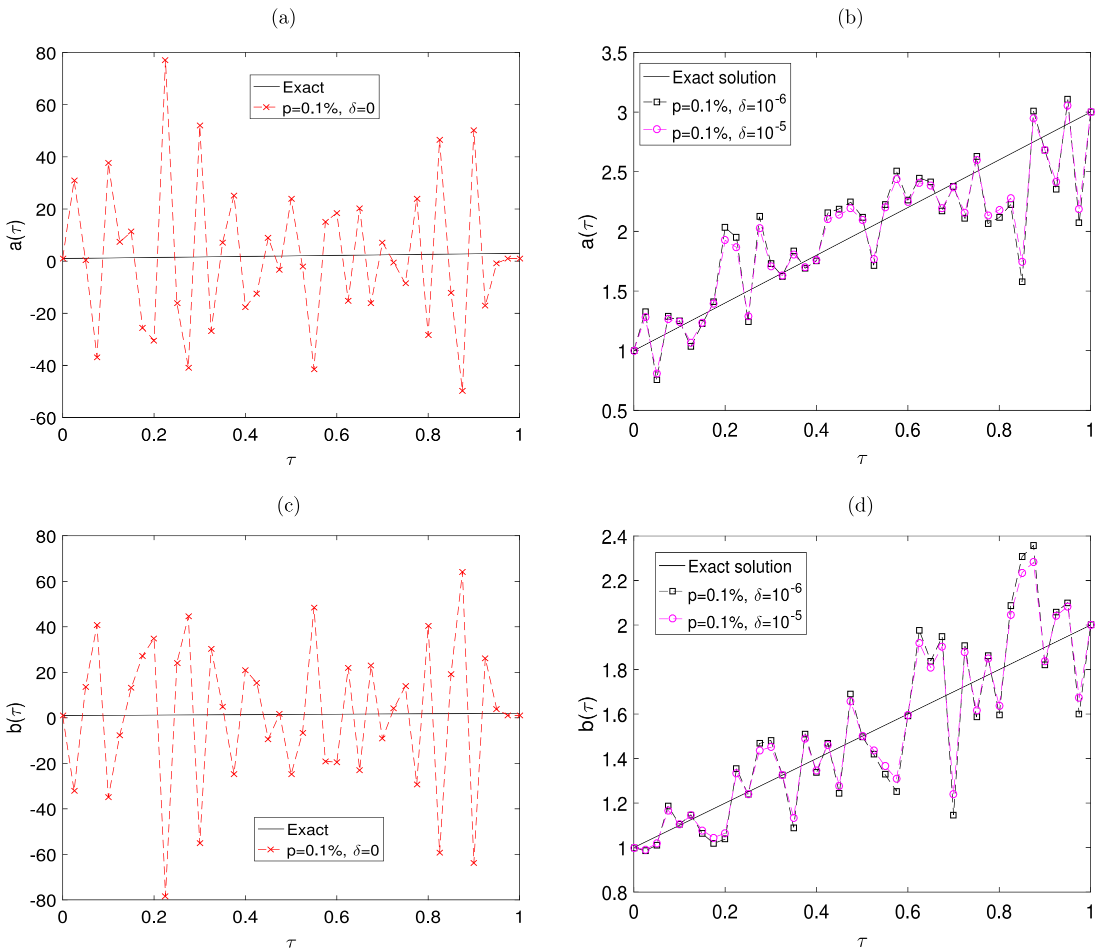

Fig. 4. The approximate and analytical (55) solutions of the potential $a(\tau)$, and the heat source $b(\tau)$, for $p = 0.1\%$, with $\delta \in \{0, 10^{-6}, 10^{-5}\}$, for Example 1.

Table 3
RMSE values, and no. of iterations, with $p \in \{0.01\%, 0.1\%\}$ noise, with $\delta \in \{0, 10^{-9}, 10^{-8}, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}\}$, for Example 1.

p	δ	RMSE(a)	RMSE(b)	Iter
0.01%	0	2.8624	3.3265	30
	10^{-9}	0.4331	0.3239	10
	10^{-8}	0.3045	0.1794	10
	10^{-7}	0.1689	0.0912	10
	10^{-6}	0.2313	0.1224	10
0.1%	0	27.8127	32.0951	50
	10^{-7}	0.4687	0.4512	20
	10^{-6}	0.3591	0.2151	20
	10^{-5}	0.3045	0.1794	20
	10^{-4}	0.4981	0.2409	20

Other details about the RMSE values ((50) and (51)), and the no. of iterations, with and without regularization are listed in Table 3. Eventually, from Figs. 2–4 and Table 3, it is observed that the MATLAB simulation results are fairly stable and accurate.

6.2. Example 2

In Example 1, smooth and linear coefficient given by Eq. (55) has been inverted. Now, we recover a discontinuous function for the potential $a(\tau)$ and a nonlinear source $b(\tau)$ terms in the inverse problem (1)–(4) subject to following input

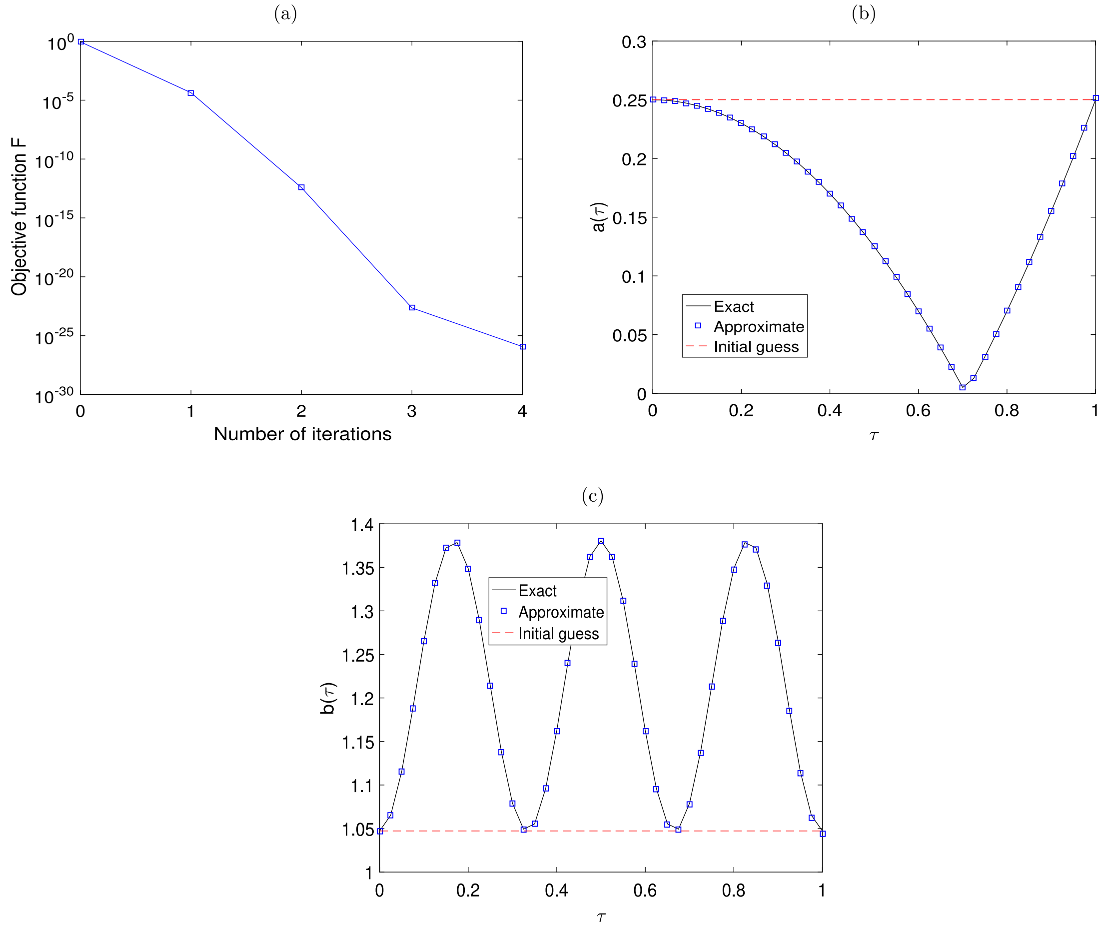

Fig. 5. (a) The unregularized form of objective function F(49) versus no. of iterations, and the approximate and analytical exact curves (57) for: (b) the potential $a(\tau)$ and (c) the source $b(\tau)$, in absence of noise and regularization, for Example 2..

data:

$$a(\tau) = \frac{1}{2}\left|\tau^2 - \frac{1}{2}\right|, \quad b(\tau) = \frac{\pi + \sin^2(3\pi\tau)}{3}, \quad \tau \in [0, 1]. \tag{57}$$

The analytical exact solution for $z(\xi, \tau)$ is the same as that has been given in (45). The other input data is kept same as it was used in Example 1 and

$$g(\xi, \tau) = \frac{3e^{e^{-\tau}\cos(\pi\xi)}\left|\tau^2 - \frac{1}{2}\right|}{-1 - 2\pi + \cos(6\pi\tau)}.$$

With the above data, the assumptions of Theorem 1 can also be verified to affirm that we have unique solution to the problem. The initial approximation for $\underline{a}$ and $\underline{b}$ are supposed to be

$$a^0(\tau_j) = a(0) = \frac{1}{4}, \quad b^0(\tau_j) = b(0) = \frac{\pi}{3}, \quad j = 1, 2, \ldots, N. \tag{58}$$

First, we use $M = N = 40$ and $p = 0$ to retrieve the unknown coefficients $a(\tau)$, $b(\tau)$ and $z(\xi, \tau)$ for the analytical input data. In Fig. 5(a), the unregularized function (49), i.e. $\delta = 0$ has been plotted versus number of iterations in absence of noise in measurement data. It can be noted that after three iterations, there is a quick decline for getting a stipulated tolerance of $O(10^{-26})$. The analytical (55) and approximate solutions to the function $a(\tau)$ and $b(\tau)$ are portrayed in Figs. 5(b) and 5(c). It is observed that the numerical outcomes are very accurate with RMSE$(a) = 1.2000$E-3 and RMSE$(b) = 3.2607$E-4.

Now, we discuss the case of noisy input data with $p \in \{0.01\%, 0.1\%\}$ by including Gaussian random noise in $h_1(\tau)$ and $h_2(\tau)$. We have also experimented with higher amounts of noise p in Eq. (54), but the results obtained were less accurate

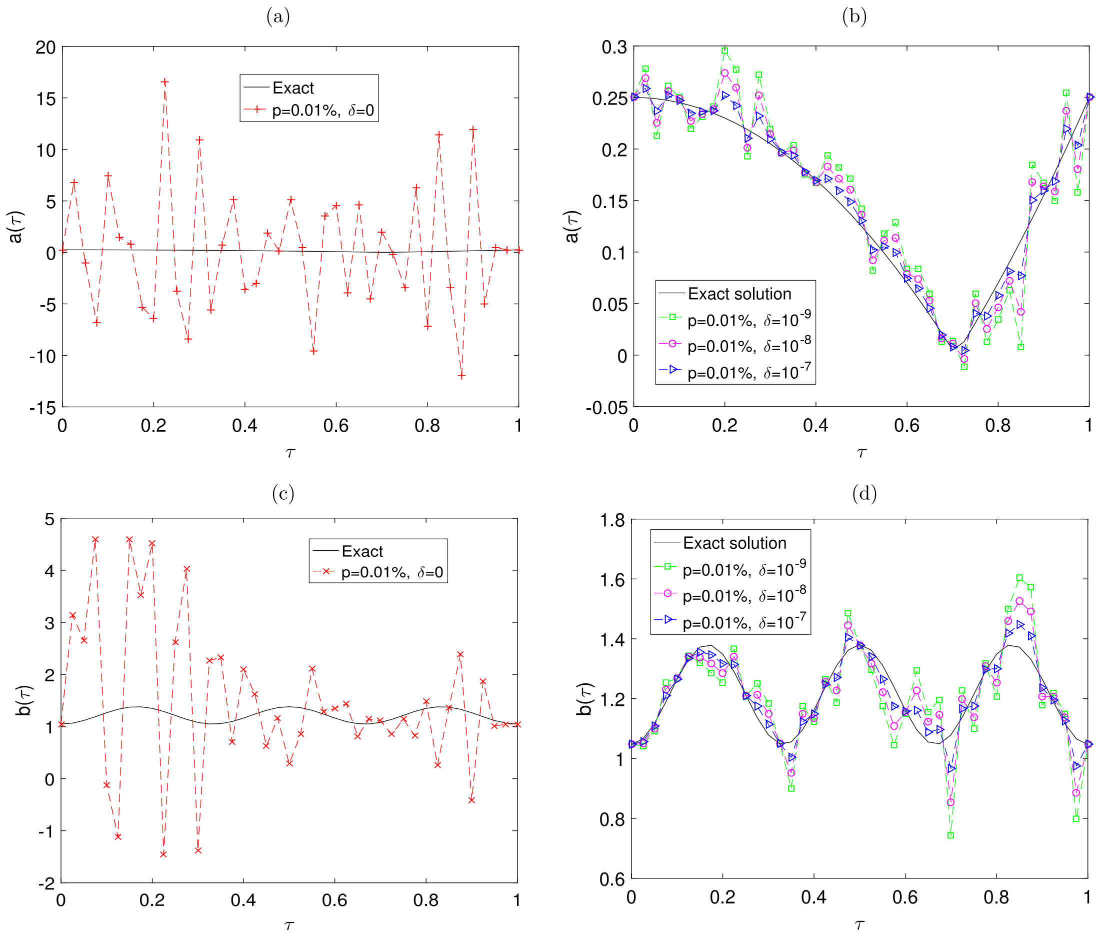

Fig. 6. The approximate and analytical (57) solutions of the potential $a(\tau)$, and the source $b(\tau)$, for $p = 0.01\%$, with $\delta \in \{0, 10^{-9}, 10^{-8}, 10^{-7}\}$, for Example 2.

and therefore they are not presented. From the previous analysis, as anticipated, without regularization, i.e., $\delta = 0$, the least-squares minimization returns an unstable solution. Hence, for restoration of the stability, we need to utilize the Tikhonov regularization approach by entering the stabilizer term in $\mathbb{F}$(49). The analytical (57) and approximate solutions for $a(\tau)$ and $b(\tau)$, with and without regularization are depicted in Figs. 6 and 7. From Figs. 6(a) and 7(a), it can be noticed that the unstable (highly oscillatory) and inaccurate results are obtained for $a(\tau)$ and $b(\tau)$, if no regularization is installed with RMSE$(a) = 6.2124$ and RMSE$(b) = 1.4852$ for $p = 0.01\%$, and RMSE$(a) = 49.9355$ and RMSE$(b) = 21.5272$ for $p = 0.1\%$. In order to stabilize the coefficients $a(\tau)$ and $b(\tau)$, we employed regularization with $\delta \in \{10^{-9}, 10^{-8}, 10^{-7}\}$ (see From Figs. 6(b) and 6(d)), obtaining RMSE$(a) \in \{0.0353, 0.0235, 0.0117\}$ and RMSE$(b) \in \{0.1236, 0.0824, 0.0412\}$ for $p = 0.01\%$, and $\delta \in \{10^{-7}, 10^{-6}, 10^{-5}\}$ (see From Figs. 7(b) and 7(d)), obtaining RMSE$(a) \in \{0.0470, 0.0411, 0.0294\}$ and RMSE$(b) \in \{0.1649, 0.1443, 0.1030\}$ for $p = 0.1\%$. The exact (45) and numerically approximated solutions for $z(\xi, \tau)$, without and with regularization, is plotted in Fig. 8, where the contribution of $\delta > 0$ in curtailing the instability of the recovered temperature can be noted. For completeness, other RMSE values and the number of iterations, without and with regularization are listed in Table 4. Overall, the computational outcomes produced by the FDM approach together with Tikhonov's regularization advocate that accurate and stable approximate solutions can be achieved for the ill-posed third order in time PDE.

7. Conclusions

The article considers the problem of determining the timewise potential and force terms in the third order in time PDE with homogeneous Neumann boundary conditions and the temperature observations. The unique solvability of the auxiliary integral problem on a sufficiently small time interval has been proved by using of the contraction principle. Also

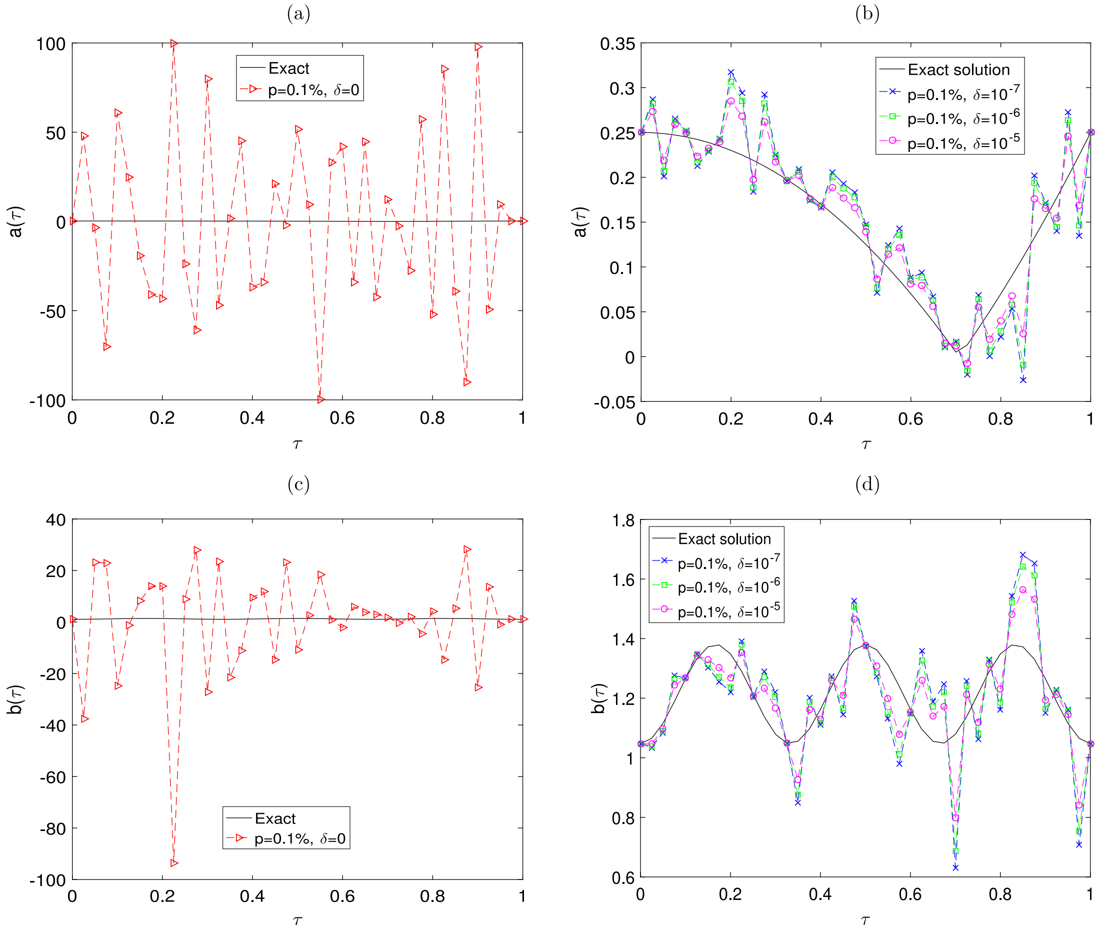

Fig. 7. The approximate and analytical (57) solutions of the potential $a(\tau)$, and the source $b(\tau)$, for $p = 0.1\%$, with $\delta \in \{0, 10^{-7}, 10^{-6}, 10^{-5}\}$, for Example 2.

Table 4
RMSE values, and no. of iterations, with $p \in \{0.01\%, 0.1\%\}$ noise, with $\delta \in \{0, 10^{-9}, 10^{-8}, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}\}$, for Example 2.

p	δ	RMSE(a)	RMSE(b)	Iter
0.01%	0	6.2124	1.4852	41
	10^{-9}	0.0353	0.1236	20
	10^{-8}	0.0235	0.0824	20
	10^{-7}	0.0117	0.0412	20
	10^{-6}	0.0210	0.0599	20
0.1%	0	49.9355	21.5272	61
	10^{-7}	0.0470	0.1649	30
	10^{-6}	0.0411	0.1443	30
	10^{-5}	0.0294	0.1030	30
	10^{-4}	0.0304	0.2165	30

the uniqueness of the solution of the inverse problem is concluded from this assertion if it exists. The proposed work is novel and has never been solved theoretically nor numerically before. The direct solver based on the FDM technique was employed. The resulting non-linear optimization problem was solved computationally by means of the MATLAB subroutine *lsqnonlin*. Since the problem under consideration was ill-posed, therefore, the Tikhonov regularization was utilized in order to tackle the stability. The numerical results for the problem show that stable and accurate approximate results have been obtained.

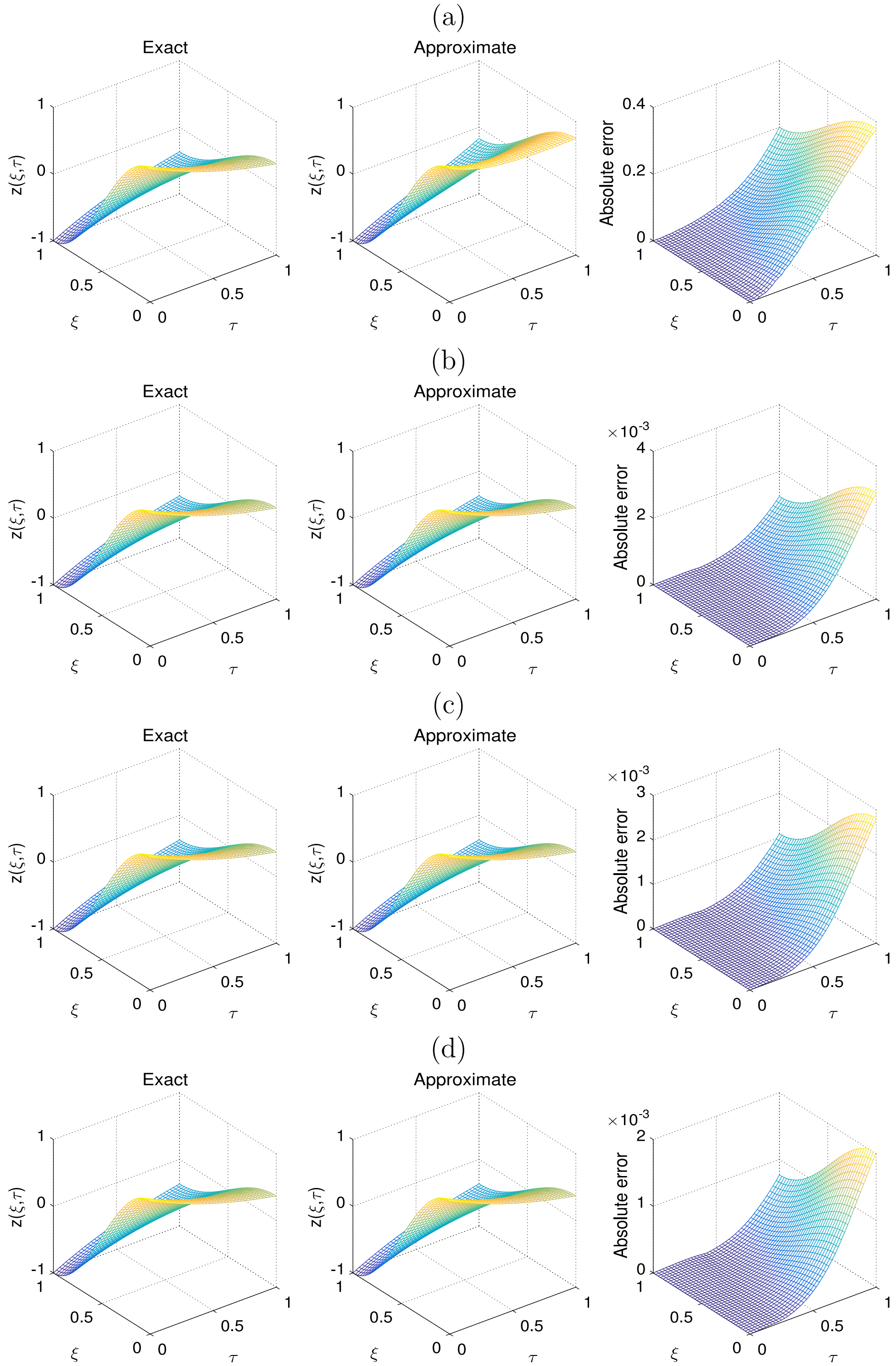

Fig. 8. The approximate and analytical (45) solutions for $z(\xi,\tau)$, with $p = 0.1\%$, $\delta \in \{0, 10^{-7}, 10^{-6}, 10^{-5}\}$ and the absolute computational error between them, for Example 2.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Arthur H Caixeta, Cavalcanti Irena Lasiecka Valéria ND. Global attractors for a third order in time nonlinear dynamics. J Differential Equations 2016;261:113–47.
[2] Conti Monica, Pata Vittorino, Quintanilla Ramon. Thermoelasticity of Moore–Gibson–Thompson type with history dependence in the temperature. Asymptotic Anal Preprint 2019;1–21.
[3] Filippo Dell'Oro, Pata Vittorino. On the Moore–Gibson–Thompson equation and its relation to linear viscoelasticity. Appl Math Optim 2017;76:641–55.
[4] Barbara Kaltenbacher, Lasiecka Irena, Pospieszalska Maria K. Well-posedness and exponential decay of the energy in the nonlinear Jordan–Moore–Gibson–Thompson equation arising in high intensity ultrasound. Math Models Methods Appl Sci 2012;22:1250035.
[5] Barbara Kaltenbacher, Nikolić Vanja. The Jordan–Moore–Gibson–Thompson Equation: Well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time. Math Models Methods Appl Sci 2019;29:2523–56.
[6] Barbara Kaltenbacher. Mathematics of nonlinear acoustics. Evol Equ Control Theory 2015;4:447–91.
[7] Pellicer Sabadí B. Wellposedness and decay rates for the Cauchy problem of the Moore–Gibson–Thompson equation arising in high intensity ultrasound. Appl Math Optim 2019;80:447–78.
[8] Gunnar Taraldsen. A generalized westervelt equation for nonlinear medical ultrasound. J Acoust Soc Am 2001;109:1329–33.
[9] Shitao Liu, Triggiani Roberto. An inverse problem for a third order PDE arising in high-intensity ultrasound: Global uniqueness and stability by one boundary measurement. J Inverse Ill-Posed Probl 2013;21:825–69.
[10] Rodrigo Lecaros, Mercado Alberto, Zamorano Sebastián. An inverse problem for Moore–Gibson–Thompson equation arising in high intensity ultrasound. 2020, arXiv preprint arXiv:2001.07673.
[11] Seyfaddin Aliyev Ziyatkhan, Mehraliyev Yashar Topush, Yusifova Elmira Haci. Inverse boundary value problem for a third-order partial differential equation with integral conditions. Bull Iran Math Soc 2021;47:1641–60.
[12] Ziyatkhan S Aliyev, Yashar T, Yusifova Elmira H. On some nonlocal inverse boundary problem for partial differential equations of third order. Turkish J Math 2021;45:1871–86.
[13] Megraliev Yashar Topush ogly, oqli Alhzade Ulvi Sahib. On solvability an inverse value problem for the equation of the third order describing the propagation of longitudinal waves in a dispersive medium with integral condition. Vestn TVGU. Seriya: Prikl Mat 2019;2:88–106.
[14] Tekin I. Inverse problem for a nonlinear third order in time partial differential equation. Math Methods Appl Sci 2021;44:9571–81.
[15] Mehraliyev YT, Shafiyeva G Kh. On an inverse boundary-value problem for a pseudoparabolic third-order equation with integral condition of the first kind. J Math Sci 2015;204:343–50.
[16] Khompysh K. Inverse problem for 1D pseudo-parabolic equation. Funct Anal Interdiscip Appl 2017;216:382–7.
[17] Ramazanova AT, Mehraliyev YT, Allahverdieva SI. On an inverse boundary value problem with non-local integral terms condition for the pseudo-parabolic equation of the fourth order. In: Differential equations and their applications in mathematical modeling, Saransk, July 9–12. 2019, p. 101–3.
[18] Serikbaev D, Tokmagambetov N. An inverse problem for the pseudo-parabolic equation for a Sturm–Liouville operator. News Natl Acad Sci Repub Kazakhstan 2019;4:122–8.
[19] Huntul MJ, Dhiman N, Tamsir M. Reconstructing an unknown potential term in the third-order pseudo-parabolic problem. Comput Appl Math 2021;40(140).
[20] Huntul MJ, Tamsir M, Dhiman N. An inverse problem of identifying the time-dependent potential in a fourth-order pseudo-parabolic equation from additional condition. Numer Methods Partial Differential Equations 2021. http://dx.doi.org/10.1002/num.22778.
[21] Huntul MJ. Determination of a time-dependent potential in the higher-order pseudo-hyperbolic problem. Inverse Probl Sci Eng 2021;29:3006–23. http://dx.doi.org/10.1080/17415977.2021.1964496.
[22] Mehraliyev YT, Shafiyeva G Kh. Determination of an unknown coefficient in the third order pseudoparabolic equation with non-self-adjoint boundary conditions. J Appl Math 2014;2014:1–7.
[23] Franklin K Moore, Gibson Walter E. Propagation of weak disturbances in a gas subject to relaxation effects. J Aerosp Sci 1960;27:117–27.
[24] Thompson PA. Compressible-fluid dynamics. McGraw-Hill; 1972.
[25] Jordan Pedro M. An analytical study of Kuznetsov's equation: diffusive solitons, shock formation, and solution bifurcation. Phys Lett A 2004;326:77–84.
[26] Bose SK, Gorain GC. Stability of the boundary stabilised internally damped wave equation $y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y')$ in a bounded domain in $\mathbb{R}^n$. Indian J Math 1998;40:1–15.
[27] Marchand Richard, McDevitt Timothy, Triggiani Roberto. An abstract semigroup approach to the third-order Moore–Gibson–Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability. Math Methods Appl Sci 2012;35:1896–929.
[28] Marta Pellicer, Solà-Morales Joan. Optimal scalar products in the Moore–Gibson–Thompson equation. Evol Equ Control Theory 2019;8(203).
[29] Mehraliyev YT. On solvability of an inverse boundary value problem for a second order elliptic equation (in Russian). Bull Tver State Univ Ser Appl Math 2011;23:25–38.
[30] Smith GD. Numerical solution of partial differential equations: finite difference methods. Oxford applied mathematics and computing science series, third ed.. 1985.
[31] Mathworks. Documentation optimization toolbox-least squares (model fitting) algorithms. 2019, available at www.mathworks.com/help/toolbox/optim/ug/brnoybu.html.
[32] Hansen PC. Analysis of discrete ill-posed problems by means of the L-curve. SIAM Rev 1992;34:561–80.
[33] Morozov VA. On the solution of functional equations by the method of regularization. Sov Math Doklady 1966;7:414–7.
[34] Dennis BH, Dulikravich GS, Yoshimura S. A finite element formulation for the determination of unknown boundary conditions for three-dimensional steady thermoelastic problems. J Heat Transfer 2004;126:110–8.