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FORMULAS FOR Q-COMBINATORIAL SIMSEK NUMBERS AND POLYNOMIALS: ANALYZING WITH COMPUTATIONAL IMPLEMENTATIONS

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This paper aims to provide new results and computational implementations for describing and analyzing the q-combinatorial Simsek numbers and polynomials of the first kind. For symbolic computation of these numbers and polynomials, some procedures and illustrations have been provided in the Wolfram programming language. In addition, some computation formulas, derivative formulas, generating functions, interpolation functions, and integral formulas pertaining to these numbers and polynomials have been derived.

1. INTRODUCTION, DEFINITIONS AND MOTIVATION

Invariably, the investigation of special numbers and polynomials arising from a counting problem and construction of their generating functions has recently become a popular topic handled by some researchers using new techniques from the perspective of combinatorics. Because combinatorics, which is an important branch of discrete mathematics deals with algebraically constructing, counting and examining discrete structures that basically meet certain criteria, has found a field of application for itself in many areas such as in particular from logic to statistical physics, from evolutionary biology to computer science (*cf.* [**1**]-[**48**]).

In this context, there have been many researchers who came up with the idea of defining a combinatorial number family that can arise in solving almost many counting problems and working on the idea of constructing their generating functions. For instance

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(among others), Simsek [**40**] has initiated a classification process for combinatorial numbers by describing his first combinatorial number family, denoted by $y_1(n,k;\lambda)$, with the following combinatorial sum:

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}j^n\lambda^j \tag{1}$$

and constructed their generating function by the following formal power series:

$$F_{y_1}(t,k;\lambda) := \frac{(\lambda\exp(t)+1)^k}{k!} = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!},$$

where exp denotes the natural exponential function (see, for details, [**40**]; and also see [**37, 41**]).

One of the special cases of the numbers $y_1(n,k;\lambda)$ is given by

$$S_2(n,k) = (-1)^k y_1(n,k;-1)$$

where

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n, \tag{2}$$

which denotes the Stirling numbers of the second kind that gives the number of ways to partition a set of n objects into k nonempty subsets, and the numbers $S_2(n,k)$ have many more such features emerging in a series of combinatorial enumeration problems (*cf.* [**5, 28, 40, 43**]).

Due to the finite sums given in (1) and (2) and others, we assume here that

$$0^n = \begin{cases} 1, & n=0 \\ 0, & n\in\mathbb{N}=\{1,2,\dots\}. \end{cases} \tag{3}$$

The numbers $y_1(n,k;\lambda)$ unify also the following combinatorial sum and identity of Golombek [**8**, Eq. (2), p. 2]:

$$B(k,n) = \sum_{j=0}^{k}\binom{k}{j}j^n = \frac{d^n}{dt^n}\left\{(\exp(t)+1)^k\right\}\Big|_{t=0} \tag{4}$$

due to the fact that

$$B(k,n) = k!y_1(n,k;1),$$

(*cf.* [**40**]).

Especially, in the literature, there exists some studies that concentrate on not only how to interpret the formula (4) combinatorically, but also whether the sum $B(k,n)$ can be given with an explicit formula in special cases. For instance, Spivey [**42**, Identity 8-Identity 10] dealt with finding an explicit formula for the sum $B(k,n)$ by finite differences, and achieved the following formulas, for some special cases of n:

$$B(k,0)=\sum_{j=0}^{k}\binom{k}{j}=2^k,$$

$$B(k,1)=\sum_{j=0}^{k}\binom{k}{j}j=k2^{k-1},$$

$$B(k,2)=\sum_{j=0}^{k}\binom{k}{j}j^2=k(k+1)2^{k-2},$$

(*cf.* [**1, 4, 42, 40, 41**]), and more generally Spivey [**42**, Identity 12] showed that

$$B(k,n)=\sum_{j=0}^{k}\binom{n}{j}j!2^{n-j}S_2\left(k,j\right).$$

Independently from Spivey, Ross [**29**, pp. 18–20, Exercises 10–12] presented a combinatorial interpretation of the sum $B(k,n)$ while dealing with a question associated with *the number of possible selections of a committee of size j, $j\leq k$, one of whom is to be determinated as chairperson from a set of k people.* Especially, Ross [**29**, p. 18, Exercise 12] provided a solution $B(k,1)=k2^{k-1}$ to the following question: *How many distinct selections result in the chairperson and the secretary being the same?*

After the numbers $y_1(n,k;\lambda)$ were brought to light by Simsek [**40**], there have also been some interesting studies examining features of those numbers. For instance, Kucukoglu and Simsek [**23**] derived ordinary and partial derivative formulas, recurrence relations and identities satisfied by the numbers $y_1(n,k;\lambda)$, and in particular they showed that the numbers $y_1(n,k;\lambda)$ are able to be related to the number of Lyndon words by virtue of the Mobius function and Mobius inversion formula. Moreover, Kucukoglu and Simsek [**22**] provided interpolation functions for the numbers $y_1(n,k;\lambda)$ by virtue of Mellin transformation and provided some plots and applications via some computational algorithms regarding these interpolation functions. In [**22**], they also presented Riemann integral and Cauchy integral representations of the numbers $y_1(n,k;\lambda)$. On the other hand, referring to a question raised by Simsek [**40**] regarding the recursive formula for $B(k,d)$, an explicit presentation, for the case when d is nonnegative integer, was provided by Xu [**48**] as in the following way:

$$B(k,0)=2^k,$$

$$B(k,d)=\frac{2^{k-d}}{m_0}\binom{k}{d}-\sum_{j=1}^{d}\frac{m_j}{m_0}B(k,d-j);\qquad d\geq 1$$

where

$$m_j=\frac{1}{d!}S_1(d,d-j);\quad j=0,1,\dots,d$$

in which $S_1(d,d-j)$ denotes the Stirling numbers of the first kind (*cf.* [**40**], [**48**]; and see also the references cited therein). The method used by Xu [**48**] was based upon the Faá di Bruno's formula by which Xu also established a formula for the numbers $y_1(n,k;\lambda)$, other than (1), as below (*cf.* [**48**]):

$$y_1(n,k;\lambda)=\sum_{j=0}^{n}\frac{S_2(n,j)}{(k-j)!}\lambda^j(\lambda+1)^{k-j}.$$

It can also be obtained from the above equation that there is a relationship between the numbers $y_1(n,k;\lambda)$ and the beta-type rational functions as follows:

$$y_1(n,k;\lambda)=\sum_{j=0}^{n}\frac{S_2(n,j)}{(k-j)!}\mathfrak{M}_{j,n}\left(\lambda\right),$$

where $\mathfrak{M}_{j,n}\left(\lambda\right)$ denotes the beta-type rational functions defined by (*cf.* [**36**]):

$$\left(\frac{\lambda}{1+\lambda}\right)^{j}\exp\left(t\left(1+\lambda\right)\right)=\sum_{n=0}^{\infty}\mathfrak{M}_{j,n}\left(\lambda\right)\frac{t^{n}}{n!}$$

such that

$$\mathfrak{M}_{j,n}\left(\lambda\right)=\lambda^{j}(1+\lambda)^{n-j};\qquad n,j\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}.$$

With the same motivation that of Xu [**48**] but via a proof different of that given by Xu [**48**], Goubi [**10**] gave an answer to the question raised by Simsek [**40**] regarding the recursive formula for $B(k,n)$ and he also presented generating functions and formulas for the numbers $y_1(n,k;\lambda)$ by calling them as "Simsek numbers". Afterwards, Goubi [**11**] constructed the generating functions for the generalization $B\left(n,k,\lambda\right)$ of Simsek numbers by the following formal power series:

$$\begin{aligned}f_n\left(\lambda,x\right) &= \sum_{k=0}^{\infty}B\left(n,k,\lambda\right)x^{k}\\ &= \sum_{j=1}^{n}\frac{j!\lambda^{j}x^{j}}{\left(1-\left(1+\lambda\right)x\right)^{j+1}}S\left(n,j\right)\end{aligned}$$

such that $B\left(n,k,\lambda\right)=k!y_1(n,k;\lambda)$ and $f_0\left(\lambda,x\right)=\frac{1}{1-(1+\lambda)x}$ (see, for details, [**11**]).

In addition to the above studies, different modifications and unifications of the numbers $y_1(n,k;\lambda)$ have also been studied by some researchers. For example (among others), very recently Oussi [**26**] introduced a degenerate version of the numbers $y_1\left(n,k;\lambda\right)$ and investigated its properties. Furthermore, Kucukoglu [**21**] introduced two q-analogues of the numbers $y_1(n,k;\lambda)$, which will be referred to here as "q-combinatorial Simsek numbers and polynomials of the first kind" (see Definition 1).

As for the main motivation of the present paper, it is aimed to investigate elaborately some features of the q-combinatorial Simsek numbers and polynomials of the first kind, which are capable of being an essential tool in solving many combinatorial type problems and modeling these problems via q-analysis identities. The other motivation and purpose of this paper is to provide computational implementations for analyzing these numbers and polynomials and to give their properties such as derivative formulas, generating functions, interpolation functions, integral formulas involving the Riemann integral, the q-integral and the p-adic q-integral.

The present study have been carried out by getting motivation especially from the wide application fields of q-calculus. Because the concept of q-calculus makes the derivative process possible without using the limit concept, and therefore has attracted the attention of many researchers due to this feature and it seems that it will continue to do so. Even a real-world problem that have not yet found a solution with classical analysis methods can occasionally be solved with the methods of q-calculus. There are also so many researchers, such as mathematicians, physicists, computer scientist and quantum machinists who tried to make new discoveries, to find out new features and to write new algorithms by introducing q-analogues of many concepts existing in the classical analysis (*cf.* [**3, 6, 7, 9, 12, 13, 15**]).

From here on, we start with reminding the following notations and definitions regarding the q-calculus:

Let $\mathbb{C}$ be the set of complex numbers, as usual. Then, a q-analogue of any positive integer k (i.e. q-integer) is defined by

$$[k]_q := \frac{1-q^k}{1-q} = 1+q+q^2+\cdots+q^{k-1}; \quad q \in \mathbb{C}\setminus\{1\}, \quad k \in \mathbb{N} \tag{5}$$

with $[0]_q = 0$, and it is clear that

$$\lim_{q\to 1} [k]_q = k \tag{6}$$

(*cf.* [**13**]; and also see [**9, 12**]).

Note that the following properties hold true:

$$[k_1+k_2]_q = [k_1]_q + q^{k_1}[k_2]_q \tag{7}$$

and

$$[k_1k_2]_q = [k_1]_q [k_2]_{q^{k_1}} \tag{8}$$

(*cf.* [**18, 24**], [**30**]-[**32**], [**34, 44**]).

A q-analogue of binomial coefficients (i.e. q-binomial coefficients), is defined by

$$\begin{bmatrix} k \\ j \end{bmatrix}_q := \frac{[k]_q!}{[j]_q!\,[k-j]_q!}; \qquad j = 0,1,\ldots,k \tag{9}$$

in which $[k]_q!$ denotes a q-analogue of the well-known factorial concept (i.e. q-factorial), given as follows:

$$[k]_q! := \begin{cases} 1, & k=0, \\ [k]_q [k-1]_q \ldots [2]_q [1]_q, & k \in \mathbb{N} \end{cases}$$

(*cf.* [**13**]; and also see [**9, 12**]).

Note that the q-binomial coefficients, given in (9), are reduced to the ordinary binomial coefficients in the limit case when $q \to 1$ (*cf.* [**9, 12, 13**]).

Besides, q-binomial coefficients have many combinatorial interpretations. One of which (among others) is that the coefficient $\begin{bmatrix} k \\ j \end{bmatrix}_q$ corresponds to the number of j-dimensional subspaces in the k-dimensional vector space $\mathbb{F}_q^k$ where $\mathbb{F}_q$ denotes a finite field with a prime power order q (see, for details, [**13**, p. 21]). Besides, the q-binomial coefficients arise in the word problems. For example, another combinatorial interpretation of the q-binomial coefficients was given by MacMahon [**25**, p. 315] as in the following expression:

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = \sum_{\sigma \in \mathfrak{S}\left(0^{k-j}1^j\right)} q^{\operatorname{inv}(\sigma)}$$

in which $\mathfrak{S}\left(0^{k-j}1^j\right)$ denotes the set of 0-1 bit strings (a sequence of binary digits) consisting of $k-j$ zeros and j ones, and also for $\sigma = \sigma_1\sigma_2\ldots\sigma_k \in \mathfrak{S}\left(0^{k-j}1^j\right)$, the number of inversions is denoted by $\operatorname{inv}(\sigma) = \#\{(m,n) : m < n \ \wedge \ \sigma_m > \sigma_n\}$. Furthermore, MacMahon [**25**, p. 318] gave a combinatorial interpretation for the q-factorial as in the following expression:

$$[k]_q! = \sum_{\sigma \in \mathfrak{S}_k} q^{\mathrm{inv}(\sigma)}$$

where $\mathfrak{S}_k$ denotes the symmetric group having exactly k disjoint cycles. For details and other applications, see [**2**] and [**25**, p. 318]; and also see the references cited therein.

1.1. The q-combinatorial Simsek numbers and polynomials of the first kind

Short a while ago, for the purpose of providing q-analogues of the numbers $y_1(n,k;\lambda)$, Kucukoglu [**21**] introduced the q-combinatorial Simsek numbers and polynomials of the first kind by the following definition:

Definition 1 (*cf.* [**21**])**.** *Let $q \in \mathbb{C} \setminus \{1\}$, $n, k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. The q-combinatorial Simsek numbers $y_{1,q}(n,k;\lambda)$ and polynomials $y_{1,q}(x;n,k;\lambda)$ of the first kind are respectively defined by the following finite sums:*

$$y_{1,q}(n,k;\lambda) = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q^n \lambda^j \tag{10}$$

and

$$y_{1,q}(x;n,k;\lambda) = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [x+j]_q^n \lambda^j. \tag{11}$$

When $n=0$, (10) and (11) are reduced to the following q-binomial formula which is known as the Gauss's q-binomial formula and the Cauchy binomial theorem:

$$\begin{aligned} [k]_q! y_{1,q}(0,k;\lambda) &= [k]_q! y_{1,q}(x;0,k;\lambda) \\ &= \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \\ &= \prod_{j=0}^{k-1} \left(1 + q^j \lambda\right) \end{aligned}$$

for which and its some applications, see [**9, 13, 39**]; and the references cited therein.

By (10) and (11), it is obvious that

$$y_{1,q}(n,k;\lambda) = y_{1,q}(0;n,k;\lambda). \tag{12}$$

Observe also that in the limit case of $q \to 1$, we have

$$\lim_{q \to 1} y_{1,q}(n,k;\lambda) = y_1(n,k;\lambda) \tag{13}$$

and

$$\lim_{q \to 1} y_{1,q}(x;n,k;\lambda) = \sum_{v=0}^{n} \binom{n}{v} x^v y_1(n-v,k;\lambda) \tag{14}$$

(*cf.* [**21**]).

Note that the q-Stirling numbers of the second kind, introduced by Carlitz [**3**], are described by the following two formulas:

$$S_{2,q}(n,k)=\frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q [k-j]_q^n$$

and

$$[x]_q^n=\sum_{k=0}^{n}q^{\binom{k}{2}}\begin{bmatrix}x\\k\end{bmatrix}_q [k]_q!S_{2,q}(n,k)$$

(*cf.* [**3, 15, 27**]).

The relationship between the numbers $y_{1,q}(n,k;\lambda)$ and the q-Stirling numbers of the second kind $S_{2,q}(n,k)$ is given as below (*cf.* [**21**]):

$$y_{1,q}(n,k;-1)=q^{\binom{k}{2}}S_{2,q}(n,k).$$

In [**21**], Kucukoglu also provided the following generating function for the numbers $y_{1,q}(n,k;\lambda)$:

$$\begin{aligned}G_{y_{1,q}}(t,k;\lambda) &:= \sum_{n=0}^{\infty}y_{1,q}(n,k;\lambda)\frac{t^n}{[n]_q!}.\\ &= \frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q \lambda^j e_q\left([j]_q t\right),\end{aligned}\tag{15}$$

where $e_q(t)$ denotes the q-exponential function (a q-analogue of the natural exponential function $\exp(t)$) defined by

$$e_q(t):=\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}$$

which converges for all $|t|<1/|1-q|$ if $|q|<1$, and for all $t\in\mathbb{C}$ if $|q|>1$ or $q=1$. Thus, for $q>0$, there exist an interval that involves t in which the series given for $e_q(t)$ converges (*cf.* [**13**, Eq. (9.7), p. 30]; and also see [**7, 9**]).

As stated in [**21**] that due to the equation (15), the series represented by $G_{y_{1,q}}(t,k;\lambda)$ converges in the region of wherever the series for q-exponential function $e_q(t)$ converges. Thus, the function $G_{y_{1,q}}(t,k;\lambda)$ converges for all $|t|<1/|1-q|$ if $|q|<1$, and for all $t\in\mathbb{C}$ if $|q|>1$ or $q=1$. Therefore, for $q>0$, there exist an interval involving t in which the series given for $G_{y_{1,q}}(t,k;\lambda)$ converges.

Note that the function $G_{y_{1,q}}(t,k;\lambda)$ satisfies the following properties (*cf.* [**21**]):

$$\lim_{q\to 1}G_{y_{1,q}}(t,k;\lambda)=F_{y_1}(t,k;\lambda)$$

and

$$D_{q,t}^n\{G_{y_{1,q}}(t,k;\lambda)\}\big|_{t=0}=y_{1,q}(n,k;\lambda);\quad (n\in\mathbb{N})$$

where $D_{q,t}^n$ denotes the n-th order q-derivative such that the q-derivative of a function $f(t)$ is defined as follows:

$$D_{q,t}\{f(t)\}=\frac{f(qt)-f(t)}{(q-1)t}$$

which, as $q \to 1$, reduces to the ordinary derivative:

$$\frac{d}{dt}\{f(t)\} = \lim_{q\to 1} D_{q,t}\{f(t)\}$$

provided that $f(t)$ is differentiable (*cf.* [**9**]; and see also [**13, 12**]).

In particular, the application of the q-derivative to the function $e_q(t)$ results as in the following:

$$D_{q,t}\{e_q(t)\} = e_q(t)$$

(*cf.* [**13**]; and see also [**9, 12**]).

Here we note that the q-derivative is also known as discrete q-derivative or Jackson derivative (*cf.* [**13**]; and see also [**9, 12**]).

Next, the content and results of the present paper have been summarized briefly as follows:

In Section 2, we give some computational implementations for symbolic computation of $y_{1,q}(x; n, k; \lambda)$ in the Wolfram Language. We also present some tables and plots, which are outputs of these implementations run for some randomly chosen special cases of $y_{1,q}(x; n, k; \lambda)$. In Section 3, we derive some properties pertaining to $y_{1,q}(n, k; \lambda)$ and $y_{1,q}(x; n, k; \lambda)$. Especially, computation formulas, derivative formulas, the Riemann integral formulas, the q-integral formulas, and the p-adic q-integral formulas have been obtained for $y_{1,q}(x; n, k; \lambda)$ and their generating functions. Moreover, we derive some differential equations via application of q-derivative operator. Besides, by applying the Mellin transform with analytic continuation to the generating functions of $y_{1,q}(n, k; \lambda)$ and $y_{1,q}(x; n, k; \lambda)$, their interpolation functions have also been constructed at negative integers. Furthermore, in Section 4, the paper has been concluded by providing some comments on the results of this paper.

2. COMPUTATIONAL IMPLEMENTATIONS OF $y_{1,q}(n, k; \lambda)$ AND $y_{1,q}(x; n, k; \lambda)$ IN THE WOLFRAM LANGUAGE

In this section, we begin to present a code snippet (see: Implementation 1) written in the Wolfram Language by the Wolfram Cloud platform [**47**] so that zero to the power of zero is defined as in the equation (3) during our computations.

Implementation 1. The following code snippet is written in the Wolfram Language so that zero to the power of zero is defined as in the equation (3). For details, see the documentations supplied by [**47**].

```
Unprotect[Power];
Power[0,0]=1;
Protect[Power];
```

Next, we provide a procedure `qinteger` of the q-integer $[k]_q$ by the Wolfram Language (see: Implementation 2).

Implementation 2. The following code, written in the Wolfram Language, involves the procedure `qinteger` which symbolically returns the q-integer $[k]_q$, by the definition that emerges from the equations (5) and (6).

```
qinteger [0, q_]:=0;
qinteger [k_?IntegerQ, q_]/; k> 0 := Which[q!=1, (1−q^k)/(1−q), q==1, k];
```

For symbolic computation of $y_{1,q}(n,k;\lambda)$ and $y_{1,q}(x;n,k;\lambda)$, we also provide a procedure `Qy1Poly` (see: Implementation 3) by implementing the equation (11) in the Wolfram Language.

Implementation 3. The following code, written in the Wolfram Language, involves the procedure `Qy1Poly` which symbolically return the q-combinatorial Simsek polynomials $y_{1,q}(x;n,k;\lambda)$ of the first kind.

```
Qy1Poly[x_, n_?IntegerQ, k_?IntegerQ, \[Lambda]_, q_] /; k>=0 && n >=0 := (1/
    QFactorial[k, q])*Sum[(q^Binomial[j,2])*QBinomial[k, j, q] *qinteger[x+j, q]^n *
    \[Lambda]^j, {j, 0, k}]
```

Remark 2. *We remark here that it is sufficient to take $x=0$ in the procedure `Qy1Poly` in order to obtain the values of the numbers $y_{1,q}(n,k;\lambda)$, due to the relationship given in* (12)*. That's why there is no need to give an additional procedure for the computation of the numbers $y_{1,q}(n,k;\lambda)$.*

By executing the procedure `Qy1Poly` in the Wolfram Cloud (*cf.* [**47**]) with the commands `TableForm` and `Plot`, (see: Implementation 4–Implementation 10), we present some tables and plots of $y_{1,q}(x;n,k;\lambda)$ obtained just for a few special cases (among others).

Implementation 4. The following code, written in the Wolfram Language, returns Table 1-Table 4, respectively.

```
TableForm[Evaluate[Table[Simplify[Qy1Poly[0, n, k, \[Lambda], 1/2]], {n, 4}, {k,
    4}]], TableHeadings −> {{"n=1", "n=2", "n=3", "n=4"}, {"k=1", "k=2", "k
   =3", "k=4"}}]
TableForm[Evaluate[Table[Simplify[Qy1Poly[0, n, k, 1, 1/2]], {n, 4}, {k, 4}]],
   TableHeadings −>{{"n=1", "n=2", "n=3", "n=4"},{"k=1", "k=2", "k=3", "k
   =4"}}]
TableForm[Evaluate[Table[Simplify[Qy1Poly[0, n, k, 2, 1/2]], {n, 4}, {k, 4}]],
   TableHeadings−>{{"n=1", "n=2", "n=3", "n=4" },{"k=1", "k=2", "k=3", "k
   =4"}}]
TableForm[Evaluate[Table[Simplify[Qy1Poly[x, n, k, \[Lambda], 1/2]], {n, 6}, {k,
    2}]], TableHeadings−>{{"n=1", "n=2", "n=3" , "n=4", "n=5", "n=6" }, {"k
   =1", "k=2"}}]
```

For example, the Implementation 4 returns Table 1-Table 4 whose entries are some values of $y_{1,q}(x;n,k;\lambda)$ in some randomly chosen special cases of their parameters.

Table 1. Several values of $y_{1,q}(x;n,k;\lambda)$ in the special cases when $x=0$, $q=\frac{1}{2}$ and $n,k\in\{1,2,3,4\}$.

	k=1	k=2	k=3	k=4
n=1	λ	$\frac{1}{2}\lambda(2+\lambda)$	$\frac{1}{12}\lambda\left(8+6\lambda+\lambda^2\right)$	$\frac{1}{168}\lambda\left(64+56\lambda+14\lambda^2+\lambda^3\right)$
n=2	λ	$\lambda+\frac{3\lambda^2}{4}$	$\frac{1}{48}\lambda\left(32+36\lambda+7\lambda^2\right)$	$\frac{\lambda\left(512+672\lambda+196\lambda^2+15\lambda^3\right)}{1344}$
n=3	λ	$\lambda+\frac{9\lambda^2}{8}$	$\frac{1}{192}\lambda\left(128+216\lambda+49\lambda^2\right)$	$\frac{\lambda\left(4096+8064\lambda+2744\lambda^2+225\lambda^3\right)}{10\,752}$
n=4	λ	$\lambda+\frac{27\lambda^2}{16}$	$\frac{1}{768}\lambda\left(512+1296\lambda+343\lambda^2\right)$	$\frac{\lambda\left(32\,768+96\,768\lambda+38\,416\lambda^2+3375\lambda^3\right)}{86\,016}$

Table 2. Several values of $y_{1,q}(x;n,k;\lambda)$ in the special cases when $x=0$, $\lambda=1$, $q=\frac{1}{2}$ and $n,k\in\{1,2,3,4\}$.

	k=1	k=2	k=3	k=4
n=1	1	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{45}{56}$
n=2	1	$\frac{7}{4}$	$\frac{25}{16}$	$\frac{465}{448}$
n=3	1	$\frac{17}{8}$	$\frac{131}{64}$	$\frac{5043}{3584}$
n=4	1	$\frac{43}{16}$	$\frac{717}{256}$	$\frac{57\,109}{28\,672}$

Table 3. Several values of $y_{1,q}(x;n,k;\lambda)$ in the special cases when $x=0$, $\lambda=2$, $q=\frac{1}{2}$, $n,k\in\{1,2,3,4\}$.

	k=1	k=2	k=3	k=4
n=1	2	4	4	$\frac{20}{7}$
n=2	2	5	$\frac{11}{2}$	$\frac{115}{28}$
n=3	2	$\frac{13}{2}$	$\frac{63}{8}$	$\frac{1375}{224}$
n=4	2	$\frac{35}{4}$	$\frac{373}{32}$	$\frac{16\,957}{1792}$

Table 4. Several values of $y_{1,q}\left(x;n,k;\lambda\right)$ in the special cases when $q=\frac{1}{2}$, $k\in\{1,2\}$ and $n\in\{1,2,3,4,5,6\}$.

	k=1	k=2
n=1	$2^{-x}\left(2\left(-1+2^{x}\right)+\left(-1+2^{1+x}\right)\lambda\right)$	$\frac{1}{3}\times 2^{-1-x}\left(2+\lambda\right)\left(4\left(-1+2^{x}\right)+\left(-1+2^{2+x}\right)\lambda\right)$
n=2	$4^{-x}\left(4\left(-1+2^{x}\right)^{2}+\left(-1+2^{1+x}\right)^{2}\lambda\right)$	$\frac{1}{3}\times 4^{-1-x}\left(32\left(-1+2^{x}\right)^{2}+12\left(-1+2^{1+x}\right)^{2}\lambda+\left(-1+2^{2+x}\right)^{2}\lambda^{2}\right)$
n=3	$8^{-x}\left(8\left(-1+2^{x}\right)^{3}+\left(-1+2^{1+x}\right)^{3}\lambda\right)$	$\frac{1}{3}\times 8^{-1-x}\left(128\left(-1+2^{x}\right)^{3}+24\left(-1+2^{1+x}\right)^{3}\lambda+\left(-1+2^{2+x}\right)^{3}\lambda^{2}\right)$
n=4	$16\left(\left(-1+2^{-x}\right)^{4}+\left(-1+2^{-1-x}\right)^{4}\lambda\right)$	$\frac{16}{3}\left(2\left(-1+2^{-x}\right)^{4}+3\left(-1+2^{-1-x}\right)^{4}\lambda+\left(-1+2^{-2-x}\right)^{4}\lambda^{2}\right)$
n=5	$32\left(\left(1-2^{-x}\right)^{5}+\left(1-2^{-1-x}\right)^{5}\lambda\right)$	$\frac{32}{3}\left(2\left(1-2^{-x}\right)^{5}+3\left(1-2^{-1-x}\right)^{5}\lambda+\left(1-2^{-2-x}\right)^{5}\lambda^{2}\right)$
n=6	$64\left(\left(-1+2^{-x}\right)^{6}+\left(-1+2^{-1-x}\right)^{6}\lambda\right)$	$\frac{64}{3}\left(2\left(-1+2^{-x}\right)^{6}+3\left(-1+2^{-1-x}\right)^{6}\lambda+\left(-1+2^{-2-x}\right)^{6}\lambda^{2}\right)$

Remark 3. *It can be observed from the Table 4 and its entries that the functions $y_{1,q}(x;n,k;\lambda)$ are polynomials of the variable λ. However, when the functions $y_{1,q}(x;n,k;\lambda)$ are evaluated with respect to the variable x, they are combination of some functions involving exponential functions and their powers. Therefore, the name to be given to the functions $y_{1,q}(x;n,k;\lambda)$ can be changed according to the variable considered during the computation.*

Implementation 5. The following code, written in the Wolfram Language, returns Figure 1.

```
Table[Plot[Evaluate[Table[Qy1Poly[x, n, k, 1, 1/2], {k, 1, 4}]], {x, −1, 1},
   AxesLabel −> {Style["x", Black, FontSize −> 16], Style[StringJoin[ToString
   [Subscript["y", "1,1/2"], StandardForm]," (x;" , ToString[n, StandardForm
   ], ", k; 1)"], Black, FontSize −> 16]}, PlotLegends −> {"k=1", "k=2", "k=3", "
   k=4"}], {n, 1, 4}]
```

As for the Implementation 5, it returns Figure 1 which illustrates some two-dimensional plots of $y_{1,q}\left(x;n,k;\lambda\right)$ in the randomly chosen special cases when $x\in[-1,1],\lambda=1$, $k\in\{1,2,3,4\}$ and $q=\frac{1}{2}$ while n changes from 1 to 4.

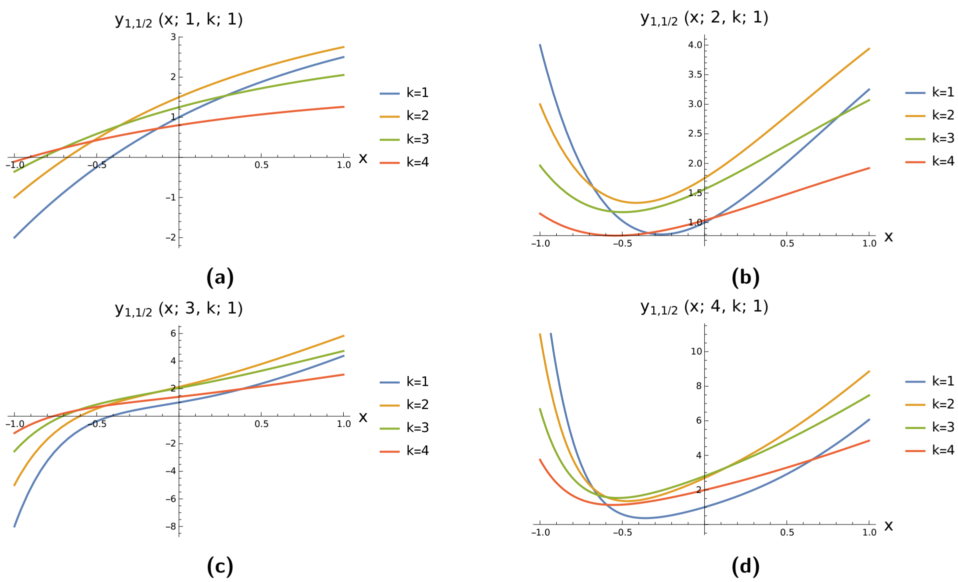


Figure 1. Plots of $y_{1,q}\,(x;n,k;\lambda)$ in the case when $x \in [-1,1], \lambda = 1$, $k \in \{1,2,3,4\}$ and $q = \frac{1}{2}$; **(a)** $n = 1$; **(b)** $n = 2$; **(c)** $n = 3$; **(d)** $n = 4$.

Implementation 6. The following code, written in the Wolfram Language, returns Figure 2.

```
Table[Plot[Evaluate[Table[Qy1Poly[x, n, k, 1, 1/2], {n, 1, 4}]], {x, −1, 1},
    AxesLabel −> {Style["x", Black, FontSize −> 16], Style[StringJoin[ToString
    [Subscript["y", "1,1/2"], StandardForm]," (x;" ,"n,", ToString[k,
    StandardForm], "; 1)"], Black, FontSize −> 16]}, PlotLegends −> {"n=1", "n
    =2", "n=3", "n=4"}], {k, 1, 4}]
```

As for the Implementation 6, it returns Figure 2 which illustrates some two-dimensional plots of $y_{1,q}\,(x;n,k;\lambda)$ in the randomly chosen special cases when $x \in [-1,1], \lambda = 1$, $n \in \{1,2,3,4\}$ and $q = \frac{1}{2}$ while k changes from 1 to 4.

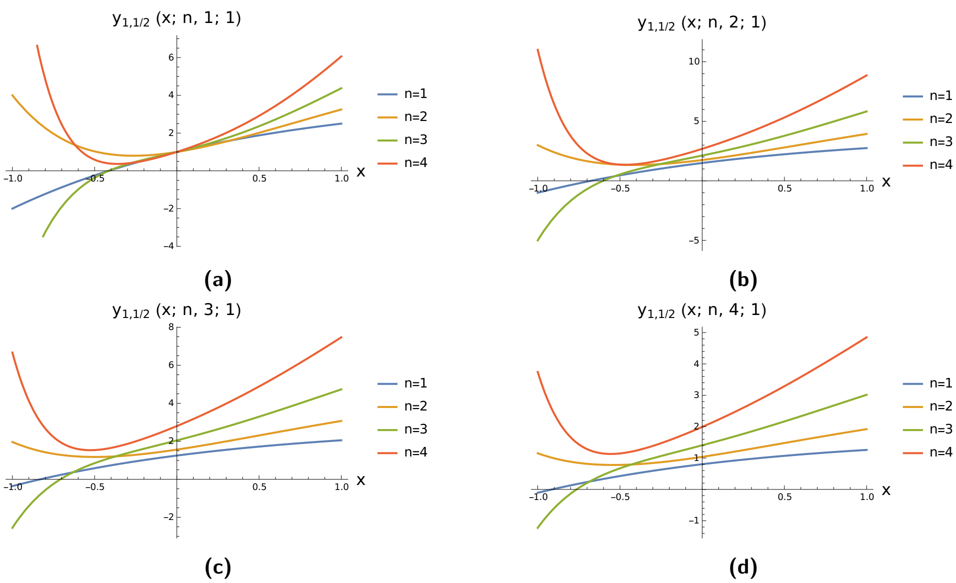


Figure 2. Plots of $y_{1,q}\left(x;n,k;\lambda\right)$ in the case when $x\in[-1,1],\lambda=1$, $n\in\{1,2,3,4\}$ and $q=\frac{1}{2}$; **(a)** $k=1$; **(b)** $k=2$; **(c)** $k=3$; **(d)** $k=4$.

Especially, Figure 1 and Figure 2 illustrate how the change of the parameters n and k affect the graphs of $y_{1,q}\left(x;n,k;\lambda\right)$ in the case when the parameters x belongs to a fixed randomly selected interval while the other parameters are the exactly same.

Implementation 7. The following code, written in the Wolfram Language, returns Figure 3.

```
Table[Plot3D[Qy1Poly[x, 5, k, \[Lambda], 1/2], {x, −1, 1}, {\[Lambda], 1, 50},
    AxesLabel −> {Style["x", Black, FontSize −> 11], Style["\[Lambda]", Black,
    FontSize −> 11], Style[StringJoin[ToString[Subscript["y", "1,1/2"],
    StandardForm], " (x;","5,",ToString[k, StandardForm], "; \[Lambda])    "],
    Black, FontSize −> 11]}], {k, 1, 4}]
```

As for the Implementation 7, it returns Figure 3 which illustrates some three-dimensional plots of $y_{1,q}\left(x;n,k;\lambda\right)$ in the randomly chosen special cases when $x\in[-1,1]$, $\lambda\in[1,50]$, $q=\frac{1}{2}$, $n=5$ and $k\in\{1,2,3,4\}$.

Figure 3 is given for the purpose of illustrating the effects of the parameter k on the three-dimensional plots of $y_{1,q}\left(x;n,k;\lambda\right)$ in the case when the parameters x and λ belongs to fixed randomly selected intervals while the other parameters are the exactly same.

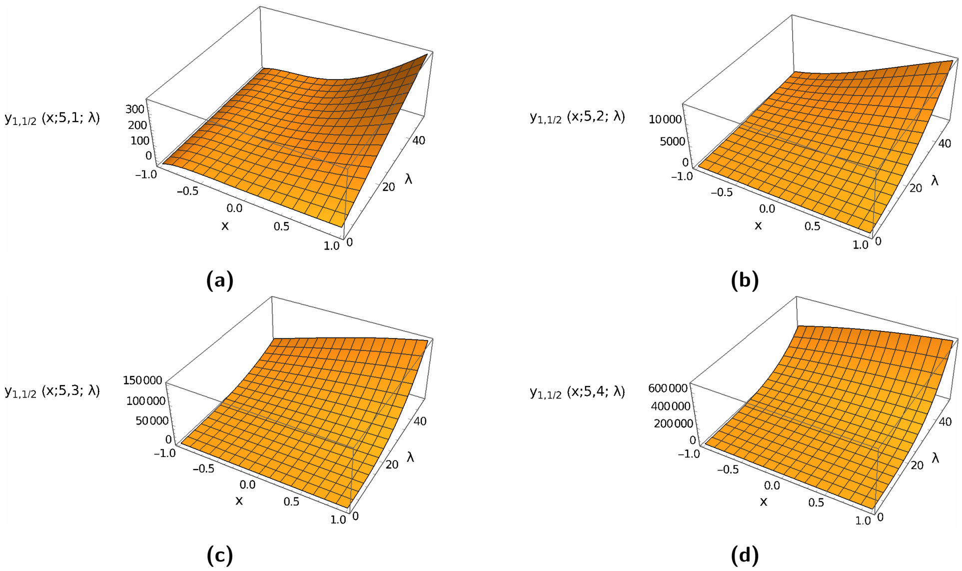


Figure 3. Plots of $y_{1,q}\,(x;n,k;\lambda)$ in the case when $x \in [-1,1], \lambda \in [1,50], q = \frac{1}{2}, n = 5$; **(a)** $k = 1$; **(b)** $k = 2$; **(c)** $k = 3$; **(d)** $k = 4$.

Implementation 8. The following code, written in the Wolfram Language, returns Figure 4.

```
ComplexPlot3D[Qy1Poly[0, 5, 3, \[Lambda], 1/2], {\[Lambda], −2−2*I, 2+2*I},
  PlotLegends −> Automatic]
```

Next, we also provide a surface plot (see: Figure 4) for illustrating graphically the zeros of the function $y_{1,q}\,(x;n,k;\lambda)$ in a randomly chosen special case (when $x = 0$, $n = 5$, $k = 3$ and $q = \frac{1}{2}$) and over the complex rectangle with corners $\lambda_{\min} = -2 - 2i$, and $\lambda_{\max} = 2 + 2i$ such that $i^2 = -1$. While plotting the Figure 4 (see: Implementation 8), we utilize from the command `ComplexPlot3D`, which is used to identify features such as zeros, poles and essential singularities (if exist). Now we briefly interpret the Figure 4 and the usage of the command `ComplexPlot3D` in the manner of the complex analysis methods: The command `ComplexPlot3D` analyzes how a color function it uses as a tool changes from $-\pi$ to π and visualizes this analysis. In the resulting image, if the color change is counterclockwise, it shows that the considered function has a zero at the point where the change occurs. On the other hand, if the color change is clockwise, it indicates that the considered function has a pole at the point where the change occurs. The multiplicity of these zeros or poles is related to the number of places where this color change occurs. For other details regarding the command `ComplexPlot3D`, the interested readers may refer to [**45**].

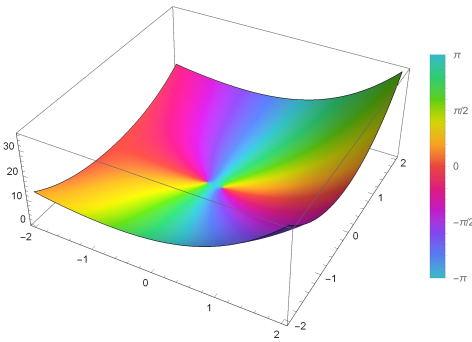
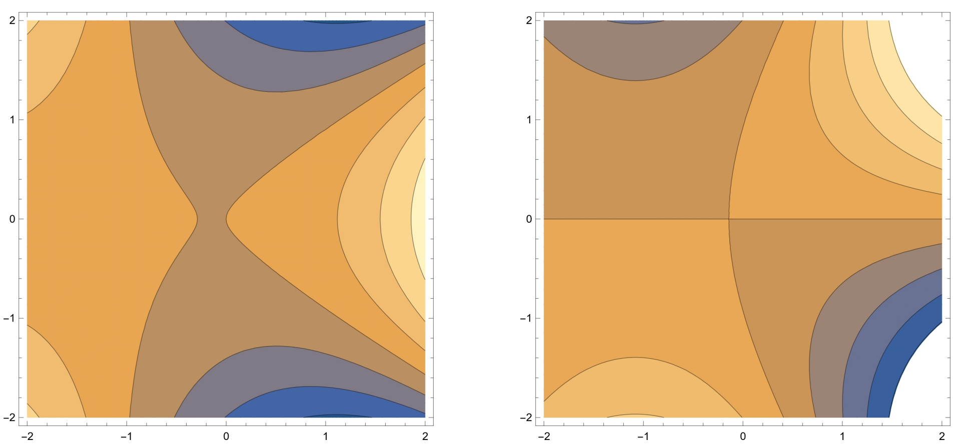


Figure 4. The ComplexPlot3D of the function $y_{1,q}\left(x;n,k;\lambda\right)$ in the case when $x=0$, $n=5$, $k=3$ and $q=\frac{1}{2}$) and over the complex rectangle with corners $\lambda_{\min}=-2-2i$, and $\lambda_{\max}=2+2i$ such that $i^2=-1$

Implementation 9. The following code, written in the Wolfram Language, returns Figure 5.

```
ContourPlot[Re[Qy1Poly[0, 5, 3, a+I*b, 1/ 2]], {a, −2, 2}, {b, −2, 2}]
ContourPlot[Im[Qy1Poly[0, 5, 3, a+I*b, 1/ 2]], {a, −2, 2}, {b, −2, 2}]
```

In addition, we also provide contour plots of the function $y_{1,q}\left(x,n,k;\lambda\right)$ (see: Figure 5). While plotting the Figure 5 (see: Implementation 9), we utilize from the command `ContourPlot`, which generates a contour plot of a function (see, for details, [**46**]).



(a) **(b)**

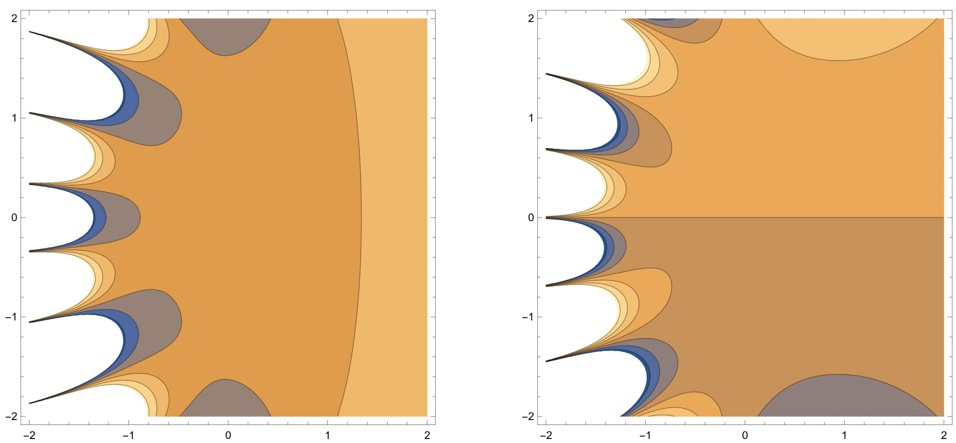
Figure 5. From left to right, the contour plots of the real and the imaginary part of the function $y_{1,q}\left(x;n,k;\lambda\right)$ in the case when $x=0$, $n=5$, $k=3$, $q=\frac{1}{2}$ and $\lambda=a+ib$ where $a\in[-2,2]$ and $b\in[-2,2]$.

Besides, by Implementation 10, we also provide contour plots of the function $y_{1,q}\left(x;n,k;\lambda\right)$

(see: Figure 6).

Implementation 10. The following code, written in the Wolfram Language, returns Figure 6.

```
ContourPlot[Re[Qy1Poly[a+I*b, 5, 3, 2, 1/2]], {a, −2, 2}, {b, −2, 2}]
ContourPlot[Im[Qy1Poly[a+I*b, 5, 3, 2, 1/2]], {a, −2, 2}, {b, −2, 2}]
```



(a) (b)

Figure 6. From left to right, the contour plots of the real and the imaginary part of the function $y_{1,q}(x;n,k;\lambda)$ in the case when $n=5$, $k=3$, $q=\frac{1}{2}$, $\lambda=2$ and $x=a+ib$ where $a\in[-2,2]$ and $b\in[-2,2]$ such that $i^2=-1$

3. DERIVATION OF SOME PROPERTIES PERTAINING TO THE NUMBERS $y_{1,q}(n,k;\lambda)$ AND THE POLYNOMIALS $y_{1,q}(x;n,k;\lambda)$

In this section, we investigate the numbers $y_{1,q}(n,k;\lambda)$, the polynomials $y_{1,q}(x;n,k;\lambda)$ and their generating functions elaborately to obtain their further properties including computation, derivative and integral formulas, and interpolation functions. Especially, by applying the Riemann integral, the q-integral and p-adic q-integral, we derive various identities for these numbers and polynomials. Moreover, by applying Mellin transform with analytic continuation to exponential generating function of the polynomials $y_{1,q}(x;n,k;\lambda)$, we construct interpolation functions of these polynomials at negative integers.

3.1. Computation formulas pertaining to the polynomials $y_{1,q}(x;n,k;\lambda)$

Here, we derive some computation formulas pertaining to the polynomials $y_{1,q}(x;n,k;\lambda)$. Using (7) in (11) implies

$$y_{1,q}(x;n,k;\lambda)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q\lambda^j\left([x]_q+q^x[j]_q\right)^n.$$

By applying the binomial theorem in the equation just above, we get

$$y_{1,q}(x;n,k;\lambda)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q\lambda^j\sum_{v=0}^{n}\binom{n}{v}[x]_q^v q^{x(n-v)}[j]_q^{n-v}$$

by which, we achieve the assertion of the following theorem:

Theorem 4. *Let $n\in\mathbb{N}_0$. Then we have*

$$y_{1,q}(x;n,k;\lambda)=\sum_{v=0}^{n}\binom{n}{v}[x]_q^v q^{x(n-v)}y_{1,q}(n-v,k;\lambda). \tag{16}$$

Remark 5. *Letting $q\to1$, the equation* (16) *turns into the equation* (14).

On the other hand, using (7) in (11) with the commutative property of addition gives

$$y_{1,q}(x;n,k;\lambda)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q\lambda^j\left([j]_q+q^j[x]_q\right)^n.$$

Applying the binomial theorem, we get

$$y_{1,q}(x;n,k;\lambda)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q\lambda^j\sum_{v=0}^{n}\binom{n}{v}[j]_q^{n-v}q^{vj}[x]_q^v,$$

which yields the assertion of the following theorem:

Theorem 6. *Let $n\in\mathbb{N}_0$. Then we have*

$$y_{1,q}(x;n,k;\lambda)=\sum_{v=0}^{n}\binom{n}{v}[x]_q^v y_{1,q}(n-v,k;\lambda q^v). \tag{17}$$

Remark 7. *By taking $q\to1$, the equation* (17) *turns into the equation* (14).

In the next, we give some results that could potentially be used to derive Raabe multiplication formulas satisfied especially by the normalized polynomials:

Replacing x by rx in (17), we obtain

$$y_{1,q}(rx;n,k;\lambda)=\sum_{v=0}^{n}\binom{n}{v}[rx]_q^v y_{1,q}(n-v,k;\lambda q^v).$$

Using (8) in the above equation, we achieve the following theorem:

Theorem 8. *Let $n \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Then we have*

$$y_{1,q}(rx; n, k; \lambda) = \sum_{v=0}^{n} \binom{n}{v} [r]_q^v [x]_{q^r}^v \, y_{1,q}(n - v, k; \lambda q^v).$$

3.2. Derivative formulas and generating functions for the q-combinatorial Simsek polynomials $y_{1,q}(x; n, k; \lambda)$ of the first kind

Here, we give some derivative formulas for the q-combinatorial Simsek polynomials $y_{1,q}(x; n, k; \lambda)$ of the first kind since one of the most important properties of a polynomial family worth investigating is derivative formula.

Let $\log(z)$ denote the principal branch of the multi-valued function $\log(z)$ with the imaginary part $\mathrm{Im}(\log(z))$ constrained by the interval $(-\pi, \pi]$.

Taking ordinary derivative of (11) with respect to x, we get

$$\frac{d}{dx}\{y_{1,q}(x; n, k; \lambda)\} = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \frac{d}{dx}\{[x + j]_q^n\}$$

which is equivalent to

$$\frac{d}{dx}\{y_{1,q}(x; n, k; \lambda)\} = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \left(\frac{n \log(q)\, q^{x+j} [x + j]_q^{n-1}}{q - 1} \right).$$

If we make an algebraic manipulation in this equation as follows:

$$\begin{aligned} \frac{d}{dx}\{y_{1,q}(x; n, k; \lambda)\} &= \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \left(\frac{n \log(q) \left(q^{x+j} - 1 + 1\right) [x + j]_q^{n-1}}{q - 1} \right) \\ &= \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \left(n \log(q) [x + j]_q^n + \frac{n \log(q)}{q - 1} [x + j]_q^{n-1} \right) \end{aligned}$$

and join the final equation with (11), then we achieve a derivative formula for $y_{1,q}(x; n, k; \lambda)$ as in the following theorem:

Theorem 9. *Let $n \in \mathbb{N}_0$. Then we have*

$$\frac{d}{dx}\{y_{1,q}(x; n, k; \lambda)\} = n \log(q) \left(y_{1,q}(x; n, k; \lambda) + \frac{y_{1,q}(x; n - 1, k; \lambda)}{q - 1} \right).$$

On the other hand, by applying the q-derivative operator $D_{q,\lambda}$ to the both-sides of the equation (10), we get

$$D_{q,\lambda}\{y_{1,q}(n, k; \lambda)\} = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q^n \, D_{q,\lambda}\{\lambda^j\}. \tag{18}$$

Since

$$D_{q,x}\{x^k\} = [k]_q x^{k-1}, \tag{19}$$

(*cf.* [**9**, p. 1012]), then (18) becomes

$$D_{q,\lambda}\{y_{1,q}(n,k;\lambda)\} = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q^{n+1} \lambda^{j-1}$$

which, by joining with (10), yields a differential equation, satisfied by $y_{1,q}(n,k;\lambda)$ with respect to λ, given as in the following theorem:

Theorem 10. *For $\lambda \neq 0$, we have*

$$D_{q,\lambda}\{y_{1,q}(n,k;\lambda)\} = \frac{1}{\lambda} y_{1,q}(n+1,k;\lambda).$$

If we apply the q-derivative operator $\lambda D_{q,\lambda}$ to (10) m times and using (19), we get

$$(\lambda D_{q,\lambda})^m \{y_{1,q}(n,k;\lambda)\} = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q^{n+m} \lambda^{j}$$

where

$$(\lambda D_{q,\lambda})^m = \lambda D_{q,\lambda}\{(\lambda D_{q,\lambda})^{m-1}\}.$$

By joining the final equation with (10), yields the following theorem:

Theorem 11. *Let $m \in \mathbb{N}$. Then we have*

$$(\lambda D_{q,\lambda})^m \{y_{1,q}(n,k;\lambda)\} = y_{1,q}(n+m,k;\lambda).$$

3.3. Exponential generating functions for the q-combinatorial Simsek polynomials $y_{1,q}(x;n,k;\lambda)$ of the first kind

Here, we introduce an exponential generating function for the polynomials $y_{1,q}(x;n,k;\lambda)$ by setting

$$\mathcal{G}_{y_{1,q}}(t,x;k;\lambda) := \sum_{n=0}^{\infty} y_{1,q}(x;n,k;\lambda)\frac{t^n}{n!}. \tag{20}$$

Substituting (11) into the right-hand side of (20) implies

$$\sum_{n=0}^{\infty} y_{1,q}(x;n,k;\lambda)\frac{t^n}{n!} = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \sum_{n=0}^{\infty} \frac{\left([x+j]_q\, t\right)^n}{n!}.$$

By using the Taylor series of the natural exponential function and combining the final equation with (20), we get the following theorem that clearly states the function $\mathcal{G}_{y_{1,q}}(t,x;k;\lambda)$ in terms of the natural exponential function:

Theorem 12. *Let $k \in \mathbb{N}_0$. Then we have*

$$\mathcal{G}_{y_{1,q}}(t,x;k;\lambda) = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \exp\left([x+j]_q\, t\right). \tag{21}$$

Observe that

$$\begin{aligned} \lim_{q\to 1} \mathcal{G}_{y_{1,q}}(t,x;k;\lambda) &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j \exp\left((x+j)\,t\right) \\ &= \frac{(\lambda\exp(t)+1)^k \exp(xt)}{k!}. \end{aligned} \tag{22}$$

Taking partial derivative of (21) with respect to t, we get

$$\frac{\partial}{\partial t}\{\mathcal{G}_{y_{1,q}}(t,x;k;\lambda)\} = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j\,[x+j]_q \exp\left([x+j]_q\, t\right),$$

and then repeating the above process n times and putting $t=0$ in the final equation, we achieve the following theorem:

Theorem 13. *Let $n \in \mathbb{N}$. Then we have*

$$\left.\frac{\partial^n}{\partial t^n}\{\mathcal{G}_{y_{1,q}}(t,x;k;\lambda)\}\right|_{t=0} = y_{1,q}(x;n,k;\lambda).$$

3.4. Identities arising from application of the Riemann integral to the polynomials $y_{1,q}(x;n,k;\lambda)$ and their generating functions

Here, we derive some identities by applying the Riemann integral to the polynomials $y_{1,q}(x;n,k;\lambda)$ and their generating functions.

Applying the Riemann integral to (11), with respect to x from 0 to t, gives

$$\int_0^t y_{1,q}(x;n,k;\lambda)\,dx = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \int_0^t [x+j]_q^n\,dx. \tag{23}$$

On the other hand, by applying the Riemann integral to the function $[x]_q^n$, we get

$$\int_0^t [x]_q^n\,dx = \frac{1}{(1-q)^n \log(q)}\left[\log\left(q^t\right) + \sum_{v=1}^{n}(-1)^v\binom{n}{v}\frac{q^{vt}-1}{v}\right] \tag{24}$$

because of the fact that

$$\lim_{v\to 0}\frac{q^{vt}-1}{v} = \log\left(q^t\right).$$

Remark 14. *For other forms of Riemann integral formulas for the functions* $[x]_q^n$ *and* $[x]_q$*, the interested readers may refer to* [**38**].

After some calculations, the integral (24) becomes accordingly as follows:

$$\int_0^t [x+j]_q^n \, dx = \frac{1}{(1-q)^n \log(q)} \left[\log\left(q^t\right) + \sum_{v=1}^{n} (-1)^v \binom{n}{v} \frac{q^{vj}\left(q^{vt}-1\right)}{v} \right]. \tag{25}$$

By combining (25) with (23), we achieve an integral formula for $y_{1,q}(x;n,k;\lambda)$ as in the following theorem:

Theorem 15. *Let* $n \in \mathbb{N}_0$ *and* $t \geq 0$*. Then we have*

$$\begin{aligned} \int_0^t y_{1,q}(x;n,k;\lambda)\, dx &= \frac{1}{(1-q)^n \log(q)\, [k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \\ &\quad \times \left[\log\left(q^t\right) + \sum_{v=1}^{n} (-1)^v \binom{n}{v} \frac{q^{vj}\left(q^{vt}-1\right)}{v} \right]. \end{aligned}$$

Applying the Riemann integral to (11), with respect to λ from 0 to t, gives

$$\int_0^t y_{1,q}(x;n,k;\lambda)\, d\lambda = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [x+j]_q^n \int_0^t \lambda^j d\lambda.$$

which yields the following theorem:

Theorem 16. *Let* $n \in \mathbb{N}_0$ *and* $t \geq 0$*. Then we have*

$$\int_0^t y_{1,q}(x;n,k;\lambda)\, d\lambda = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [x+j]_q^n \frac{t^{j+1}}{j+1}. \tag{26}$$

Setting $x = 0$ in (26) and joining the final equation with (12), we get the following corollary:

Corollary 17. *Let* $n \in \mathbb{N}_0$ *and* $t \geq 0$*. Then we have*

$$\int_0^t y_{1,q}(n,k;\lambda)\, d\lambda = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q^n \frac{t^{j+1}}{j+1}. \tag{27}$$

Remark 18. *By taking* $q \to 1$ *in the equation* (27) *and joining the final equation with* (13)*, we arrive at the Lemma* 4.1 *of* [**22**, p. 51].

Applying the Riemann integral to (21), with respect to t from 0 to u, gives

$$\int_0^u \mathcal{G}_{y_{1,q}}(t,x;k;\lambda)\,dt = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \int_0^u \exp\left([x+j]_q\, t\right) dt$$

which yields the following theorem:

Theorem 19. *Let $k \in \mathbb{N}_0$. Then we have*

$$\int_0^u \mathcal{G}_{y_{1,q}}(t,x;k;\lambda)\,dt = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{\lambda^j\left(\exp\left([x+j]_q\, u\right)-1\right)}{[x+j]_q}.$$

Remark 20. *By taking $q \to 1$ in the Theorem* 19 *and joining the final equation with* (22)*, we arrive at the following (presumably) new integral formula:*

$$\int_0^u \left((\lambda \exp(t)+1)^k \exp(xt)\right) dt = \sum_{j=0}^{k} \binom{k}{j} \frac{\lambda^j\left(\exp((x+j)\,u)-1\right)}{x+j}.$$

Applying the Riemann integral to (21), with respect to λ from 0 to u, gives

$$\int_0^u \mathcal{G}_{y_{1,q}}(t,x;k;\lambda)\,d\lambda = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \exp\left([x+j]_q\, t\right) \int_0^u \lambda^j d\lambda$$

which yields the following theorem:

Theorem 21. *Let $k \in \mathbb{N}_0$. Then we have*

$$\int_0^u \mathcal{G}_{y_{1,q}}(t,x;k;\lambda)\,d\lambda = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{u^{j+1}\exp\left([x+j]_q\, t\right)}{j+1}.$$

3.5. Identities arising from application of the q-integral to the polynomials $y_{1,q}(x;n,k;\lambda)$

Here, we derive some identities by applying the q-integral to the polynomials $y_{1,q}(x;n,k;\lambda)$.

Recall that the q-integral of a function $f(x)$ is defined by

$$\int_0^t f(x)d_qx = t\,(1-q)\sum_{j=0}^{\infty} f(tq^j)q^j$$

as long as the infinite sum converges (*cf.* [**9**]; and see the references cited therein).

For example, the q-integral of the function x^n is given by

$$\int_0^t x^n d_q x = \frac{t^{n+1}}{[n+1]_q}, \tag{28}$$

(*cf.* [**9**, p. 1012]).

Here we note that the q-integral is also known as discrete q-integral or Jackson integral (*cf.* [**7, 9**]).

Applying the q-integral to (11), with respect to λ, gives

$$\int_0^t y_{1,q}(x;n,k;\lambda)\, d_q\lambda = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [x+j]_q^n \int_0^t \lambda^j d_q\lambda.$$

Using (28) in the above equation, we achieve the following theorem:

Theorem 22.

$$\int_0^t y_{1,q}(x;n,k;\lambda)\, d_q\lambda = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [x+j]_q^n \frac{t^{j+1}}{[j+1]_q}.$$

3.6. Identities arising from application of the p-adic q-integral formulas to the polynomials $y_{1,q}(x;n,k;\lambda)$ and their generating functions

Here, we derive p-adic q-integral formulas pertaining to the polynomials $y_{1,q}(x;n,k;\lambda)$ and their generating functions.

Let $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{K}$ be a field with a complete valuation and $C^1(\mathbb{Z}_p \to \mathbb{K})$ be a set of continuous differentiable functions. The p-adic q-integral of a function $g \in C^1(\mathbb{Z}_p \to \mathbb{K})$ is defined by

$$\int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} q^x g(x)$$

where $\mu_q(x)$ denotes the q-Haar distribution given by

$$\mu_q(x) = \mu_q(x + p^N\mathbb{Z}_p) = \frac{q^x}{[p^N]_q},$$

(*cf.* [**17**]).

There are so many researchers who apply the p-adic q-integral to some special functions and polynomials for discovering and investigating their new features. For instance (among others), see [**16**]-[**19**] and the other references cited herein.

Recall that the q-Bernoulli polynomials are defined by (*cf.* [**18**]):

$$\beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{j=1}^{n} (-1)^j \binom{n}{j} q^{xj} \frac{j+1}{[j+1]_q},$$

whose p-adic q-integral representation is given by

$$\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_q(y), \tag{29}$$

(*cf.* [**16, 17, 19**]).

If we apply the p-adic q-integral to (11) with respect to x, we get

$$\int_{\mathbb{Z}_p} y_{1,q}(x;n,k;\lambda) \, d\mu_q(x) = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \int_{\mathbb{Z}_p} [x+j]_q^n \, d\mu_q(x)$$

and combining it with (29), we achieve a formula for p-adic q-integral of the polynomials $y_{1,q}(x;n,k;\lambda)$, given by the following theorem:

Theorem 23. *Let $n,k \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} y_{1,q}(x;n,k;\lambda) \, d\mu_q(x) = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \beta_{n,q}(j). \tag{30}$$

Furthermore, by applying p-adic q-integral to (17) with respect to x, we get

$$\int_{\mathbb{Z}_p} y_{1,q}(x;n,k;\lambda) \, d\mu_q(x) = \sum_{v=0}^{n} \binom{n}{v} y_{1,q}(n-v,k;\lambda q^v) \int_{\mathbb{Z}_p} [x]_q^v \, d\mu_q(x)$$

which, combining with (29), yields another formula for p-adic q-integral of the polynomials $y_{1,q}(x;n,k;\lambda)$, as stated in the following theorem:

Theorem 24. *Let $n,k \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} y_{1,q}(x;n,k;\lambda) \, d\mu_q(x) = \sum_{v=0}^{n} \binom{n}{v} y_{1,q}(n-v,k;\lambda q^v) \, \beta_{v,q} \tag{31}$$

where $\beta_{v,q}$ denotes the q-Bernoulli numbers defined by $\beta_{v,q} := \beta_{v,q}(0)$.

Combining the right-hand sides of the equations (30) and (31), we also arrive at the following corollary:

Corollary 25. *Let $n, k \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{n} \binom{n}{v} y_{1,q}\left(n-v, k; \lambda q^{v}\right) \beta_{v,q} = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \beta_{n,q}(j).$$

Recall that

$$\int_{\mathbb{Z}_p} \exp(tx)\, d\mu_q(x) = \left(\frac{\log(q) + t}{q \exp(t) - 1} \right) \frac{q-1}{\log(q)}, \tag{32}$$

(*cf.* [**14, 20**] and see the references cited therein).

If we apply the p-adic q-integral to (21), with respect to t, we get

$$\int_{\mathbb{Z}_p} \mathcal{G}_{y_{1,q}}(t, x; k; \lambda)\, d\mu_q(t) = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \int_{\mathbb{Z}_p} \exp\left([x+j]_q\, t\right) d\mu_q(t),$$

which, combining with (32), yields a formula for p-adic q-integral of the function $\mathcal{G}_{y_{1,q}}(t, x; k; \lambda)$, given by the following theorem:

Theorem 26. *Let $k \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \mathcal{G}_{y_{1,q}}(t, x; k; \lambda)\, d\mu_q(t) = \frac{q-1}{[k]_q! \log(q)} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \left(\frac{\log(q) + [x+j]_q}{q \exp\left([x+j]_q\right) - 1} \right).$$

3.7. Application of the Mellin transform with analytic continuation to the generating functions for the polynomials $y_{1,q}(x; n, k; \lambda)$

Here, by using the same method of Simsek [**33, 35**] and Ozden et al. [**27**], we construct interpolation functions for the q-combinatorial Simsek polynomials $y_{1,q}(x; n, k; \lambda)$ of the first kind.

Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. By applying the Mellin transform with analytic continuation to the equation (21), we have

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \mathcal{G}_{y_{1,q}}(-t, x; k; \lambda)\, dt &= \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j \frac{1}{\Gamma(s)} \\ &\quad \times \int_{0}^{\infty} t^{s-1} \exp\left(-[x+j]_q\, t\right) dt \\ &= \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{\lambda^j}{[x+j]_q^s}. \end{aligned}$$

By analytic continuation and special values of the above integral representation, we define a complex valued function $z_q(s, x; k; \lambda)$ by the following definition:

Definition 27. *For $s \in \mathbb{C}$ and $k \in \mathbb{N}$. Then, the function $z_q(s, x; k; \lambda)$ is defined by*

$$z_q(s, x; k; \lambda) := \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{\lambda^j}{[x+j]_q^s}.$$

By substituting $s = -n$; $(n \in \mathbb{N})$ into the above equation, we obtain

$$\begin{aligned} z_q(-n, x; k; \lambda) &= \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \lambda^j [x+j]_q^n \\ &= y_{1,q}(x; n, k; \lambda). \end{aligned}$$

Therefore, we arrive at the following result:

Theorem 28. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then we have*

$$z_q(-n, x; k; \lambda) = y_{1,q}(x; n, k; \lambda).$$

Remark 29. *Theorem* 28 *shows that the function $z_q(s, x; k; \lambda)$ interpolates the q-combinatorial Simsek polynomials $y_{1,q}(x; n, k; \lambda)$ of the first kind at negative integers. If we also set*

$$z_q(s; k; \lambda) := z_q(s, 0; k; \lambda) = \frac{1}{[k]_q!} \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{\lambda^j}{[j]_q^s},$$

then the q-combinatorial Simsek numbers $y_{1,q}(n, k; \lambda)$ of the first kind are interpolated by the function $z_q(s; k; \lambda)$ at negative integers. Namely, for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have

$$z_q(-n; k; \lambda) = y_{1,q}(n, k; \lambda).$$

4. CONCLUSION

In this study, the numbers $y_{1,q}(n, k; \lambda)$ and the polynomials $y_{1,q}(x; n, k; \lambda)$ has been analyzed with their some properties comprising derivative formulas, computation formulas, generating functions, interpolation functions, the Riemann integral formulas, the q-integral formulas and p-adic q-integral formulas. Along with these properties, some computational implementations and procedures have been presented, and by implementing these procedures, the q-combinatorial Simsek numbers and polynomials of the first kind have been illustrated with their tables and plots. Therefore, the results of this paper are of potential to find an application in all areas of computational and applied mathematics especially dealing with discrete mathematics. Hence, this paper might get attention of all the researchers working on the related areas.

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