Turkish Journal of Mathematics

Volume 45 | Number 5 Article 32

1-1-2021

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İREM KÜÇÜKOĞLU

YILMAZ ŞİMŞEK

Recommended Citation

KÜÇÜKOĞLU, İ, & ŞİMŞEK, Y (2021). Computational identities for extensions of some families of special numbers and polynomials. *Turkish Journal of Mathematics 45* (5): 2341-2365. https://doi.org/10.3906/mat-2101-83

TÜBİTAK

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2341 – 2365
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doi:10.3906/mat-2101-83

Computational identities for extensions of some families of special numbers and polynomials

İrem KÜÇÜKOĞLU[1,*], **Yılmaz ŞİMŞEK**[2]
[1]Department of Engineering Fundamental Sciences, Faculty of Engineering, Alanya Alaaddin Keykubat University, Antalya, Turkey
[2]Department of Mathematics, Faculty of Science, University of Akdeniz, Antalya, Turkey

Received: 19.01.2021 • **Accepted/Published Online:** 26.04.2021 • **Final Version:** 16.09.2021

Abstract: The main purpose of this paper is to obtain computational identities and formulas for a certain class of combinatorial-type numbers and polynomials. By the aid of the generating function technique, we derive a recurrence relation and an infinite series involving the aforementioned class of combinatorial-type numbers. By applying the Riemann integral to the combinatorial-type polynomials with multivariables, we present some integral formulas for these polynomials, including the Bernoulli numbers of the second kind. By the implementation of the p-adic integral approach to the combinatorial-type polynomials with multivariables, we also obtain formulas for the Volkenborn integral and the fermionic p-adic integral of these polynomials. Furthermore, we provide an approximation for the combinatorial-type numbers with the aid of the Stirling's approximation for factorials. By coding some of our results in Mathematica using the Wolfram programming language, we also provide some numerical evaluations and illustrations on the combinatorial-type numbers and their Stirling's approximation with table and figures. We also give some remarks and observations on the combinatorial-type numbers together with their relationships to other well-known special numbers and polynomials. As a result of these observations, we derive some computation formulas containing the Dirichlet series involving the Möbius function, the Bernoulli numbers, the Catalan numbers, the Stirling numbers, the Apostol–Bernoulli numbers, the Apostol–Euler numbers, the Apostol–Genocchi numbers and some kinds of combinatorial numbers. Besides, some inequalities for the combinatorial-type numbers are presented. Finally, we conclude this paper by briefly overviewing the results with their potential applications.

Key words: Generating functions, p-adic integrals, Catalan numbers, special numbers and polynomials, Combinatorial numbers, Stirling approximation

1. Introduction

To the present day, many significant techniques has been used to derive computation formulas, recurrence relations, and derivative formulas for special numbers and polynomials. Among these techniques, generating functions and p-adic integration are commonly used ones, which have many applications in mathematics, physics and engineering (*cf.* [1]-[52]).

The main motivation of this paper is to derive new computational formulas and relations, including a certain class of combinatorial-type numbers and polynomials, by using the methods of not only generating functions but also the p-adic integration methods. In addition to that, the other motivation of this paper is

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*Correspondence: irem.kucukoglu@alanya.edu.tr

2010 *AMS Mathematics Subject Classification:* 05A10, 05A15, 05A19, 11A25, 11B37, 11B68, 11B73, 11B83, 11M99, 11S23, 11S80, 11Y99, 26C05, 30B50, 33C20, 33F05, 65D20.

to give an approximation for these numbers and polynomials by the help of the Stirling's approximation for factorials.

Throughout this paper, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \dots\} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}\,.$$

Also, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the set of real numbers and the set of complex numbers.

In this paper, we are motivated to derive formulas and obtain relations for the combinatorial-type numbers $V_n(\lambda)$ and the combinatorial-type polynomials $V_n(x;\lambda)$ whose generating functions were respectively constructed by Kucukoglu et al. [30] as follows:

$$F_V(t,\lambda) = \frac{1-\lambda+\sqrt{(\lambda-1)^2+8\lambda^2 t}}{2\lambda^2 t} = \sum_{n=0}^{\infty} V_n(\lambda)\, t^n \tag{1.1}$$

and

$$F_V(t,x;\lambda) = F_V(t,\lambda)\,(1+t)^{\frac{x}{2}} = \sum_{n=0}^{\infty} V_n(x;\lambda)\, t^n \tag{1.2}$$

where $0 < \left|\frac{\lambda^2 t}{(\lambda-1)^2}\right| \leq \frac{1}{8}$.

It should be noted here that the generating function $F_V(t,\lambda)$ satisfies the following algebraic equation:

$$\lambda^2 t F_V^2(t,\lambda) + (\lambda-1)\, F_V(t,\lambda) - 2 = 0, \tag{1.3}$$

and for $n \in \mathbb{N}$, the equation (1.3) gives us a recurrence relation for the numbers $V_n(\lambda)$ as follows:

$$V_n(\lambda) = \frac{n\lambda^2}{1-\lambda} \sum_{j=0}^{n-1} V_j(\lambda)\, V_{n-j-1}(\lambda) \tag{1.4}$$

with the initial condition $V_0(\lambda) = \frac{2}{\lambda-1}$ (*cf.* [30]).

For $n \in \mathbb{N}_0$, a relation between the polynomials $V_n(x;\lambda)$ and the numbers $V_n(\lambda)$ is given by

$$V_n(x;\lambda) = \sum_{j=0}^{n} \frac{\left(\frac{x}{2}\right)_j}{j!} V_{n-j}(\lambda)\,, \tag{1.5}$$

(*cf.* [30]), where $(u)_j$ denotes the falling factorial given by

$$(u)_j = u\,(u-1)\dots(u-j+1)\,; \qquad (j \in \mathbb{N})$$

such that $(u)_0 = 1$ (*cf.* [5, 50]).

By (1.4) and (1.5), first few values of the numbers $V_n(\lambda)$ and the polynomials $V_n(x;\lambda)$ are respectively computed as follows:

$$V_0(\lambda) = 2\,(\lambda-1)^{-1}\,, \qquad V_1(\lambda) = -4\lambda^2(\lambda-1)^{-3},$$
$$V_2(\lambda) = 16\lambda^4(\lambda-1)^{-5}, \qquad V_3(\lambda) = -80\lambda^6(\lambda-1)^{-7},$$

and

$$V_0\left(x;\lambda\right)=2(\lambda-1)^{-1},\qquad V_1\left(x;\lambda\right)=(\lambda-1)^{-1}x-4\lambda^2(\lambda-1)^{-3},$$
$$V_2\left(x;\lambda\right)=x^2(\lambda-1)^{-1}-\left(4\lambda^2-4\lambda+2\right)(\lambda-1)^{-3}x+16\lambda^4(\lambda-1)^{-5},$$

and so on. For details about the computation formulas for the numbers $V_n(\lambda)$, the polynomials $V_n\left(x;\lambda\right)$ and their computational algorithms, the readers may refer to [30].

Other definitions and notations, required to be able to obtain new results in the rest of this paper, are given as follows:

The generating function for the Apostol–Bernoulli numbers, $\mathcal{B}_n(\lambda)$, is given by

$$\frac{t}{\lambda e^t-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(\lambda)\frac{t^n}{n!};\qquad (\lambda\in\mathbb{C}) \tag{1.6}$$

which converges when $|t|<2\pi$ if $\lambda=1$ and $|t|<|\ln(\lambda)|$ if $\lambda\neq1$. By using (1.6), first few values of these numbers are computed as follows:

$$\mathcal{B}_0\left(\lambda\right)=0,\qquad \mathcal{B}_1\left(\lambda\right)=(\lambda-1)^{-1},$$
$$\mathcal{B}_2\left(\lambda\right)=-2\lambda(\lambda-1)^{-2},\qquad \mathcal{B}_3\left(\lambda\right)=3\lambda\left(\lambda+1\right)(\lambda-1)^{-3},$$

and so on (*cf.* [1, 31, 48, 50, 51]; and the references cited therein).

Substituting $\lambda=1$ into (1.6) yields the generating function of the Bernoulli numbers B_n of the first kind, namely:

$$B_n=\mathcal{B}_n(1),$$

such that $B_{2n+1}=0$ for $n\in\mathbb{N}$ (*cf.* [1, 31, 48, 50, 51]; and the references cited therein).

The generating function for the Apostol–Euler numbers, $\mathcal{E}_n(\lambda)$ is given by:

$$\frac{2}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\frac{t^n}{n!}, \tag{1.7}$$

which converges when $|t|<\pi$ if $\lambda=1$ and $|t|<|\ln(-\lambda)|$ if $\lambda\neq1$. By using (1.7), first few values of these numbers are computed as follows:

$$\mathcal{E}_0\left(\lambda\right)=2\left(\lambda+1\right)^{-1},\qquad \mathcal{E}_1\left(\lambda\right)=-2\lambda(\lambda+1)^{-2},$$
$$\mathcal{E}_2\left(\lambda\right)=2\lambda\left(\lambda-1\right)(\lambda+1)^{-3},\qquad \mathcal{E}_3\left(\lambda\right)=-2\lambda\left(\lambda^2-4\lambda+1\right)(\lambda+1)^{-4},$$

and so on (*cf.* [6, 11, 31, 47, 48, 50, 51]; and the references cited therein).

Substituting $\lambda=1$ into (1.7) yields the generating function of the Euler numbers E_n of the first kind, namely:

$$E_n=\mathcal{E}_n(1), \tag{1.8}$$

such that $E_{2n}=0$ for $n\in\mathbb{N}$ (*cf.* [6, 11, 31, 47, 48, 50, 51]; and the references cited therein).

The generating function for the Apostol-Genocchi numbers, $\mathcal{G}_n(\lambda)$, is given by:

$$\frac{2t}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{G}_n(\lambda)\frac{t^n}{n!} \tag{1.9}$$

which converges when $|t|<\pi$ if $\lambda=1$ and $|t|<|\ln(-\lambda)|$ if $\lambda\neq 1$ (*cf.* [31, 48, 50, 51]; and the references cited therein).

For $n\in\mathbb{N}$ and under the suitable conditions, the well-known relations among the Apostol–Bernoulli numbers, the Apostol–Bernoulli numbers and the Apostol–Genocchi numbers are given as follows:

$$\mathcal{B}_n(\lambda) = -\frac{n\mathcal{E}_{n-1}(-\lambda)}{2}, \tag{1.10}$$

$$\mathcal{E}_n(\lambda) = \frac{\mathcal{G}_{n+1}(-\lambda)}{n+1}, \tag{1.11}$$

$$\mathcal{G}_n(\lambda) = -2\mathcal{B}_n(-\lambda) \tag{1.12}$$

(*cf.* [31, 48, 50, 51]; and the references cited therein).

The numbers $Y_n(\lambda)$ are defined by the following generating function:

$$\frac{2}{\lambda(1+\lambda t)-1}=\sum_{n=0}^{\infty}Y_n(\lambda)\frac{t^n}{n!} \tag{1.13}$$

(*cf.* [41]; and see also [29, 52]), and these numbers are computed by the following explicit formula:

$$Y_n(\lambda)=2(-1)^n\frac{n!\lambda^{2n}}{(\lambda-1)^{n+1}}, \tag{1.14}$$

(*cf.* [41, Theorem 14]).

By (1.14), first few values of the numbers $Y_n(\lambda)$ are given as follows:

$$Y_0(\lambda)=2(\lambda-1)^{-1}, \qquad Y_1(\lambda)=-2\lambda^2(\lambda-1)^{-2}, \qquad Y_2(\lambda)=4\lambda^4(\lambda-1)^{-3},$$
$$Y_3(\lambda)=-12\lambda^6(\lambda-1)^{-4}, \qquad Y_4(\lambda)=48\lambda^8(\lambda-1)^{-5},$$

and so on (*cf.* [41, 52]).

Modification of the numbers $Y_n(\lambda)$ have also been studied by Choi [7].

Notice that the following relationship exists among the numbers $Y_n(\lambda)$, the Apostol–Euler numbers and the Stirling numbers of the first kind (*cf.* [52, Eq.(33)]):

$$Y_m(-\lambda)=(-1)^{m+1}\lambda^m\sum_{n=0}^{m}\mathcal{E}_n(\lambda)S_1(m,n), \tag{1.15}$$

where $m\in\mathbb{N}_0$ and $S_1(m,n)$ stands for the Stirling numbers of the first kind defined by

$$(u)_m=\sum_{n=0}^{m}S_1(m,n)u^n, \tag{1.16}$$

(*cf.* [5, 50]).

In addition, the following relationship exists among the numbers $Y_n(\lambda)$, the Apostol–Bernoulli numbers and the Stirling numbers of the second kind (*cf.* [41, Eq.(2.23)]):

$$\mathcal{B}_m(\lambda) = \frac{m}{2} \sum_{n=0}^{m-1} \lambda^{-n} Y_n(\lambda) S_2(m-1, n), \tag{1.17}$$

where $m \in \mathbb{N}$ and $S_2(m,n)$ stands for the Stirling numbers of the second kind defined by

$$u^m = \sum_{n=0}^{m} S_2(m,n)(u)_n,$$

(*cf.* [5, 6, 8, 37, 50]).

The Catalan numbers, C_n, are defined by the following explicit formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n}; \qquad (n \in \mathbb{N}_0) \tag{1.18}$$

whose ordinary generating function is given by

$$\frac{1-\sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} C_n t^n,$$

where $0 < |t| \leq \frac{1}{4}$ (*cf.* [5], [9, pp. 96-106], [28, pp. 109-110], [35]).

The Catalan numbers inherently emerges in the solution of some kinds of combinatorial enumeration problems such as the Euler's polygon problem, the Ballot problems, the Dyck Path and etc. (see, for details, *cf.* [5], [9, pp. 96-106], [15], [28, pp. 109-110], [35]). As for the studies used the techniques of generating function in order to derive identities for the Catalan numbers, the interested readers may refer to the papers [10, 12, 15, 18, 23, 25, 27, 28, 30, 33, 35, 38].

The generating functions for the combinatorial numbers $y_6(n,k;\lambda,v)$ is defined by the generalized hypergeometric function as follows:

$$\frac{1}{k!}\, {}_vF_{v-1}\left[\begin{matrix} -k, -k, ..., -k \\ 1, 1, ..., 1 \end{matrix}\ ; (-1)^v \lambda e^t\right] = \sum_{n=0}^{\infty} y_6(n,k;\lambda,v)\frac{t^n}{n!},$$

such that the numbers $y_6(n,k;\lambda,v)$ are computed by

$$y_6(n,k;\lambda,v) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}^v j^n \lambda^j, \tag{1.19}$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) and $n,k,v \in \mathbb{N}_0$ (*cf.* [43, p. 1347]).

The Bernoulli numbers of the second kind (or so-called Cauchy numbers), $b_n(0)$, are defined by

$$b_n(0) = \int_0^1 (u)_n \,\mathrm{d}u, \tag{1.20}$$

(see, for details, [26, 32], [34, p. 116]).

The generalization of Vandermonde's convolution is given by

$$\binom{x+\sum\limits_{k=1}^{v} y_k}{n} = \sum_{k_0+k_1+\cdots+k_v=n} \binom{x}{k_0}\binom{y_1}{k_1}\cdots\binom{y_v}{k_v}, \tag{1.21}$$

(*cf.* [14, Exercise 62, p.248]). Note that in the special case when $v=1$ and $y_1=y$, (1.21) is reduced to the following well-known Chu–Vandermonde identity:

$$\binom{x+y}{n} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(x)_k\,(y)_{n-k}\,, \tag{1.22}$$

(*cf.* [8, 16, 29, 44, 45]).

The outline of this paper is summarized below:

In Section 2, by the aid of the generating function technique, we derive a recurrence relation and an infinite series involving the numbers $V_n(\lambda)$.

In Section 3, by applying the Riemann integral to the combinatorial-type polynomials with multivariables, we present some integral formulas for these polynomials, including the Bernoulli numbers of the second kind.

In Section 4, by implementing the p-adic integrals approach to the combinatorial-type polynomials with the Volkenborn and fermionic p-adic integrals, we obtain some formulas for the Volkenborn and fermionic p-adic integrals of the combinatorial-type polynomials with multivariables. The obtained formulas include some kinds of special numbers and polynomials.

In Section 5, we provide an approximation for the numbers $V_n(\lambda)$ by the aid of the Stirling's approximation for factorials. By coding some of our results in Mathematica using the Wolfram programming language, we also provide some numerical evaluations and illustrations on the numbers $V_n(\lambda)$ and their Stirling's approximation with table and figures.

In Section 6, we give some remarks and observations on the combinatorial-type numbers together with the relationships of these numbers to other well-known special numbers and polynomials. As a result of these observations, we derive some computation formulas containing the Dirichlet series involving the Möbius function, the Bernoulli numbers, the Catalan numbers, the Stirling numbers, the Apostol–Bernoulli numbers, the Apostol–Euler numbers, the Apostol–Genocchi numbers and some kinds of combinatorial numbers. Besides, we present some inequalities for the numbers $V_n(\lambda)$.

In Section 7, we conclude this paper by briefly overviewing its results with their potential applications in not only mathematics but also computational science and engineering.

2. Several computational identities containing the numbers $V_n(\lambda)$

In this section, we derive a recurrence relation for the numbers $V_n(\lambda)$ by the aid of their generating function. We also give an infinite series, containing the numbers $V_n(\lambda)$, with its evalution.

Theorem 2.1 *Let $n\in\mathbb{N}$ and $\lambda\neq 1$. Then we have*

$$V_n(\lambda) = -\sum_{j=0}^{n}\frac{\left(\frac{1}{2}\right)_j 8^j}{j!}\left(\frac{\lambda}{\lambda-1}\right)^{2j} V_{n-j}(\lambda)\,, \tag{2.1}$$

with the initial condition $V_0(\lambda) = \frac{2}{\lambda-1}$.

Proof By the modification of (1.1), we have

$$\frac{-4}{(1-\lambda)\left(1+\sqrt{1+\frac{8\lambda^2}{(\lambda-1)^2}t}\right)} = \sum_{n=0}^{\infty} V_n(\lambda)\, t^n. \tag{2.2}$$

By making cross multiplication in the equation just above, we obtain

$$\frac{4}{\lambda-1} = \sum_{n=0}^{\infty} V_n(\lambda)\, t^n + \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n 8^n}{n!}\left(\frac{\lambda}{\lambda-1}\right)^{2n} t^n \sum_{n=0}^{\infty} V_n(\lambda)\, t^n. \tag{2.3}$$

By applying the Cauchy product rule to the right-hand side of the above equation, we get

$$\frac{4}{\lambda-1} = \sum_{n=0}^{\infty} V_n(\lambda)\, t^n + \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{\left(\frac{1}{2}\right)_j 8^j}{j!}\left(\frac{\lambda}{\lambda-1}\right)^{2j} V_{n-j}(\lambda)\, t^n.$$

Therefore, we arrive at the assertion of Theorem 2.1. □

By combining the following equality (*cf.* [28]):

$$\frac{\left(\frac{1}{2}\right)_n}{n!} = \frac{(-1)^{n+1}}{2^{2n}(2n-1)}\binom{2n}{n}; \qquad (n \in \mathbb{N}_0)$$

with the equation (2.1), we get

$$V_n(\lambda) = \sum_{j=0}^{n} (-1)^j \frac{2^j}{2j-1}\binom{2j}{j}\left(\frac{\lambda}{\lambda-1}\right)^{2j} V_{n-j}(\lambda).$$

Thus, by using (1.18) in the equation just above, we arrive at the following corollary:

Corollary 2.2 *Let* $n \in \mathbb{N}$ *and* $\lambda \neq 1$. *Then we have*

$$V_n(\lambda) = \sum_{j=0}^{n} (-1)^j \frac{2^j (j+1)}{2j-1}\left(\frac{\lambda}{\lambda-1}\right)^{2j} C_j V_{n-j}(\lambda). \tag{2.4}$$

By substituting $t = \frac{(\lambda-1)^2}{8\lambda^2}$ into (1.1) with $\lambda > 1$, we get an infinite series containing the numbers $V_n(\lambda)$ by the following theorem:

Theorem 2.3 *Let* $\lambda > 1$. *Then we have*

$$\sum_{n=0}^{\infty} \frac{V_n(\lambda)}{2^n}\left(\frac{\lambda-1}{2\lambda}\right)^{2n} = \frac{4\left(\sqrt{2}-1\right)}{\lambda-1}. \tag{2.5}$$

It should be noted here that there exists a relationship between the numbers $V_n(\lambda)$ and the Catalan numbers for $n \in \mathbb{N}_0$ as follows (*cf.* [30]):

$$V_n(\lambda) = (-1)^n C_n \frac{2^{n+1}\lambda^{2n}}{(\lambda-1)^{2n+1}}. \tag{2.6}$$

The combination of (2.6) with (2.5) yields known infinite series containing the Catalan numbers with its evaluation as in the following corollary:

Corollary 2.4

$$\sum_{n=0}^{\infty} (-1)^n \frac{C_n}{2^{2n-1}} = 4\left(\sqrt{2}-1\right). \tag{2.7}$$

3. Application of the Riemann integral to the combinatorial-type polynomials with multivariables

In this section, we present some formulas resulting from the application of the Riemann integral to the combinatorial-type polynomials with multivariables.

Theorem 3.1 *Let $n \in \mathbb{N}_0$ and $\lambda \neq 1$. Then we have*

$$\underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{(v+1)\text{-times}} V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) \mathrm{d}x\mathrm{d}y_1\ldots\mathrm{d}y_v = \sum_{j=0}^{n} \sum_{k_0+k_1+\cdots+k_v=j} \prod_{m=0}^{v} \frac{b_{k_m}(0)}{k_m!} V_{n-j}(\lambda), \tag{3.1}$$

where $b_{k_m}(0)$ denotes the ${k_m}^{th}$ Bernoulli numbers of the second kind.

Proof Replacing x by $2\left(x+\sum\limits_{k=1}^{v} y_k\right)$ in (1.5), we have

$$V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) = \sum_{j=0}^{n} \frac{V_{n-j}(\lambda)}{j!} \left(x+\sum_{k=1}^{v} y_k\right)_j. \tag{3.2}$$

By combining (1.21) with (3.2), we get

$$V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) = \sum_{j=0}^{n} \sum_{k_0+k_1+\cdots+k_v=j} \binom{x}{k_0}\binom{y_1}{k_1}\cdots\binom{y_v}{k_v} V_{n-j}(\lambda). \tag{3.3}$$

By applying the Riemann integral, $(v+1)$-times, to (3.3), we get

$$\begin{aligned}\underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{(v+1)\text{-times}} V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) \mathrm{d}x\mathrm{d}y_1\ldots\mathrm{d}y_v &= \sum_{j=0}^{n} \sum_{k_0+k_1+\cdots+k_v=j} \underbrace{\int_0^1 \int_0^1 \cdots \int_0^1}_{(v+1)\text{-times}} \binom{x}{k_0}\binom{y_1}{k_1}\cdots\binom{y_v}{k_v} \\ &\quad \times V_{n-j}(\lambda)\,\mathrm{d}x\mathrm{d}y_1\ldots\mathrm{d}y_v.\end{aligned}$$

By combining (1.20) with the above equation, we arrive at the assertion of Theorem 3.1. □

Remark 3.2 *Substituting* $v = 1$ *and* $y_1 = y$ *into* (3.1)*, we have*

$$\int_0^1 \int_0^1 V_n(2x + 2y; \lambda)\, \mathrm{d}x\mathrm{d}y = \sum_{j=0}^{n} \sum_{k_0+k_1=j} \frac{b_{k_0}(0)\, b_{k_1}(0)}{k_0! k_1!} V_{n-j}(\lambda).$$

Remark 3.3 *For the other Riemann integral representations of the polynomials* $V_n(x; \lambda)$*, the interested readers may glance at the paper* [30]*.*

4. Applications of the Volkenborn integral and the fermionic p-adic integral to the combinatorial-type polynomials with multivariables

In this section, we begin with reminding the definitions and notations in association with the p-adic integrals technique that forms the basis of the derivation of the results of this section. We then obtain some formulas with the aid of this technique.

Let $\mu_1(x)$ be the Haar distribution on the set $\mathbb{Z}_p$ of p-adic integers and the function h be uniformly differentiable function on $\mathbb{Z}_p$. Then, the Volkenborn (bosonic p-adic) integral of the function h on $\mathbb{Z}_p$ is defined by

$$\int_{\mathbb{Z}_p} h(x)\, d\mu_1(x) = \lim_{N\to\infty} p^{-N} \sum_{x=0}^{p^N-1} h(x),$$

(*cf.* [36]; see also [21, 22, 24, 42, 44]).

In order to give an instance, the Volkenborn integral of the falling factorial $(x)_n$ is given as follows:

$$\int_{\mathbb{Z}_p} (x)_n\, d\mu_1(x) = \frac{(-1)^n n!}{n+1}, \tag{4.1}$$

(*cf.* [36]; and see also [17, 21, 22, 24, 44, 45]).

Let $\mu_{-1}(x) = (-1)^x$. Then, the fermionic p-adic integral of the function h on $\mathbb{Z}_p$ is defined by

$$\int_{\mathbb{Z}_p} h(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x h(x),$$

(*cf.* [21, 22, 42, 44]).

For an instance, the fermionic p-adic integral of the falling factorial $(x)_n$ is given as

$$\int_{\mathbb{Z}_p} (x)_n\, d\mu_{-1}(x) = \frac{(-1)^n n!}{2^n}, \tag{4.2}$$

(*cf.* [19]; and see also [22, 24, 44, 45]).

By using the fermionic p-adic integral of $\left(\frac{x}{2}\right)_n$, Kim [23, Theorem 3, p.497] gave the fermionic p-adic integral representation of the Catalan numbers by the following formula:

$$C_n = \frac{(-1)^n 2^{2n}}{n!} \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_{-1}(x), \tag{4.3}$$

(*cf.* [23, 27, 39, 44]).

4.1. Formulas resulting from the Volkenborn integral

Here, we present some formulas resulting from the application of the Volkenborn integral to the combinatorial-type polynomials with multivariables by using the similar techniques that of [20] and [46].

Theorem 4.1 *Let* $n \in \mathbb{N}_0$ *and* $\lambda \neq 1$*. Then we have*

$$\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-}times} V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) d\mu_1(x)\, d\mu_1(y_1)\ldots d\mu_1(y_v) \tag{4.4}$$

$$= \sum_{j=0}^{n} \sum_{k_0+k_1+\cdots+k_v=j} \frac{(-1)^{\sum_{m=0}^{v} k_m}}{\prod_{m=0}^{v}(k_m+1)} V_{n-j}(\lambda).$$

Proof By applying $(v+1)$-times the Volkenborn integral to (3.3), we get

$$\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-times}} V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) d\mu_1(x)\, d\mu_1(y_1)\ldots d\mu_1(y_v)$$

$$= \sum_{j=0}^{n} \sum_{k_0+k_1+\cdots+k_v=j} \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-times}} \binom{x}{k_0}\binom{y_1}{k_1}\cdots\binom{y_v}{k_v} V_{n-j}(\lambda)\, d\mu_1(x)\, d\mu_1(y_1)\ldots d\mu_1(y_v).$$

By combining (4.1) with the above equation, we arrive at the assertion of Theorem 4.1. □

Remark 4.2 *Substituting* $v=1$ *and* $y_1 = y$ *into* (4.4)*, we have*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} V_n(2x+2y;\lambda)\, d\mu_1(x)\, d\mu_1(y) = \sum_{j=0}^{n}\sum_{k_0+k_1=j} \frac{(-1)^{k_0+k_1}}{(k_0+1)(k_1+1)} V_{n-j}(\lambda).$$

By combining (2.6) with (4.4), we get the following corollary:

Corollary 4.3 *Let* $n \in \mathbb{N}_0$ *and* $\lambda \neq 1$*. Then we have*

$$\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-}times} V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) d\mu_1(x)\, d\mu_1(y_1)\ldots d\mu_1(y_v) \tag{4.5}$$

$$= \sum_{j=0}^{n} \sum_{k_0+k_1+\cdots+k_v=j} \frac{(-1)^{n-j+\sum_{m=0}^{v} k_m}\, 2^{n-j+1}\lambda^{2(n-j)}}{(\lambda-1)^{2(n-j)+1}\prod_{m=0}^{v}(k_m+1)} C_{n-j}.$$

Remark 4.4 *Since*

$$D_n = \frac{(-1)^n n!}{n+1},$$

where D_n *stands for the so-called Daehee numbers* (*cf.* [17]), *Theorem* 4.1 *can be written in the following form:*

$$\begin{aligned}\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-times}} & V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) d\mu_1(x)\, d\mu_1(y_1)\ldots d\mu_1(y_v) \\ &= \sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\prod_{m=0}^{v}\frac{D_{k_m}}{k_m!}V_{n-j}(\lambda).\end{aligned} \tag{4.6}$$

Thus, by using the right-hand sides of (4.4), (4.5) *and* (4.6), *we have the following combinatorial sums that are equal to each other:*

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\frac{(-1)^{\sum\limits_{m=0}^{v}k_m}}{\prod\limits_{m=0}^{v}(k_m+1)}V_{n-j}(\lambda) &= \sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\prod_{m=0}^{v}\frac{D_{k_m}}{k_m!}V_{n-j}(\lambda) \\ &= \sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\frac{(-1)^{n-j+\sum\limits_{m=0}^{v}k_m}2^{n-j+1}\lambda^{2(n-j)}}{(\lambda-1)^{2(n-j)+1}\prod\limits_{m=0}^{v}(k_m+1)}C_{n-j}\end{aligned} \tag{4.7}$$

where $\lambda \neq 1$.

4.2. Formulas resulting from the fermionic p-adic integral

Here, we present some formulas resulting from the application of the fermionic p-adic integral to the combinatorial-type polynomials with multi-variables.

Theorem 4.5 *Let* $n \in \mathbb{N}_0$ *and* $\lambda \neq 1$. *Then we have*

$$\begin{aligned}\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-times}} & V_n\left(2\left(x+\sum_{k=1}^{v} y_k\right);\lambda\right) d\mu_{-1}(x)\, d\mu_{-1}(y_1)\ldots d\mu_{-1}(y_v) \\ &= \sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\left(-\frac{1}{2}\right)^{\sum\limits_{m=0}^{v}k_m}V_{n-j}(\lambda).\end{aligned} \tag{4.8}$$

Proof By applying $(v+1)$-times the fermionic p-adic integral to (3.3), we get

$$\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-times}} V_n\left(2\left(x+\sum_{k=1}^{v}y_k\right);\lambda\right)d\mu_{-1}(x)\,d\mu_{-1}(y_1)\ldots d\mu_{-1}(y_v)$$

$$=\sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-times}}\binom{x}{k_0}\binom{y_1}{k_1}\cdots\binom{y_v}{k_v}$$

$$\times V_{n-j}(\lambda)\,d\mu_{-1}(x)\,d\mu_{-1}(y_1)\ldots d\mu_{-1}(y_v).$$

By combining (4.2) with the above equation, we arrive at the assertion of Theorem 4.5. □

Remark 4.6 *Substituting* $v=1$ *and* $y_1=y$ *into* (4.8), *we have*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}V_n(2x+2y;\lambda)\,d\mu_{-1}(x)\,d\mu_{-1}(y)=\sum_{j=0}^{n}\sum_{k_0+k_1=j}\left(-\frac{1}{2}\right)^{k_0+k_1}V_{n-j}(\lambda).$$

By combining (2.6) with (4.8), we get the following corollary:

Corollary 4.7 *Let* $n\in\mathbb{N}_0$ *and* $\lambda\neq 1$. *Then we have*

$$\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-times}} V_n\left(2\left(x+\sum_{k=1}^{v}y_k\right);\lambda\right)d\mu_{-1}(x)\,d\mu_{-1}(y_1)\ldots d\mu_{-1}(y_v) \tag{4.9}$$

$$=\sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\frac{(-1)^{n-j+\sum_{m=0}^{v}k_m}\lambda^{2(n-j)}}{2^{j-n-1+\sum_{m=0}^{v}k_m}(\lambda-1)^{2(n-j)+1}}C_{n-j}.$$

Remark 4.8 *Since*

$$Ch_n=\frac{(-1)^n n!}{2^n}, \tag{4.10}$$

where Ch_n *stands for the so-called Changhee numbers* (*cf.* [19]), *Theorem* 4.5 *can be written in the following form:*

$$\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{(v+1)\text{-times}} V_n\left(2\left(x+\sum_{k=1}^{v}y_k\right);\lambda\right)d\mu_{-1}(x)\,d\mu_{-1}(y_1)\ldots d\mu_{-1}(y_v) \tag{4.11}$$

$$=\sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\prod_{m=0}^{v}\frac{Ch_{k_m}}{k_m!}V_{n-j}(\lambda).$$

Thus, by using the right-hand sides of (4.4), (4.5) and (4.6), we have the following combinatorial sums that are equal to each other:

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\left(-\frac{1}{2}\right)^{\sum\limits_{m=0}^{v}k_m}V_{n-j}(\lambda) &= \sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\prod_{m=0}^{v}\frac{Ch_{k_m}}{k_m!}V_{n-j}(\lambda) \qquad (4.12)\\ &=\sum_{j=0}^{n}\sum_{k_0+k_1+\cdots+k_v=j}\frac{(-1)^{n-j+\sum\limits_{m=0}^{v}k_m}\lambda^{2(n-j)}}{2^{j-n-1+\sum\limits_{m=0}^{v}k_m}(\lambda-1)^{2(n-j)+1}}C_{n-j}\end{aligned}$$

where $\lambda \neq 1$*. The above sums indicate that the combinatorial-type polynomials are related to the Boole-type and the Peters-type numbers and polynomials. For details about the Boole-type and the Peters-type numbers and polynomials, see* [42].

Remark 4.9 *In addition to the above multiple* p*-adic integrals, in* [30]*, Kucukoglu et al. also gave the Volkenborn and the fermionic* p*-adic integrals of the polynomials* $V_n(x;\lambda)$ *respectively by the following formulas:*

$$\int_{\mathbb{Z}_p} V_n(x;\lambda)\,d\mu_1(x) = \sum_{j=0}^{n}\sum_{k=0}^{j}\frac{V_{n-j}(\lambda)\,S_1(j,k)B_k}{j!2^k}, \qquad (4.13)$$

and

$$\int_{\mathbb{Z}_p} V_n(x;\lambda)\,d\mu_{-1}(x) = \sum_{j=0}^{n}\sum_{k=0}^{j}\frac{V_{n-j}(\lambda)\,S_1(j,k)E_k}{j!2^k},$$

(see, for details, [30]).

5. Numerical evaluations of the numbers $V_n(\lambda)$ via the Stirling's approximation for factorials

In this section, we provide evaluations on the approximation for the numbers $V_n(\lambda)$ with the aid of the Stirling's approximation for factorials. The evaluations and computations in this section help readers to analyze approximation for the numbers $V_n(\lambda)$ and their Stirling's approximation and motivate the readers to use in their future studies.

By using Stirling's approximation for factorials, we first investigate approximate values of the numbers $V_n(\lambda)$ when its index is sufficiently large.

The Stirling's approximation for factorials is given by (*cf.* [28], [50]):

$$n! \approx \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}. \qquad (5.1)$$

Using (5.1), Koshy [28, p. 110] gave approximation for the Catalan numbers as follows:

$$C_n \approx \pi^{-\frac{1}{2}}4^n n^{-\frac{3}{2}}. \qquad (5.2)$$

Applying (5.2) and (5.1) to the equation (2.6) yields an approximation for the numbers $V_n(\lambda)$ given by the following theorem:

Theorem 5.1 *Let* n *be sufficiently large and* $\lambda \neq 1$. *Then we have*

$$V_n(\lambda) \approx V_n^{\star}(\lambda), \tag{5.3}$$

where

$$V_n^{\star}(\lambda) = (-1)^n \pi^{-\frac{1}{2}} 2^{3n+1} n^{-\frac{3}{2}} \frac{\lambda^{2n}}{(\lambda-1)^{2n+1}}.$$

Remark 5.2 *Theorem 5.1 states that the numbers* $V_n(\lambda)$ *grow asymptotically as (5.3). That is, the quotient* $V_n^{\star}(\lambda)/V_n(\lambda)$ *tends towards to* 1 *as* $n \to \infty$.

By coding some of our results in Mathematica* using the Wolfram programming language, we provide some numerical evaluations and illustrations on the approximation for the numbers $V_n(\lambda)$ and their Stirling's approximation with table and figures.

By Figure 1, we have a plot illustrating the convergence tendency of the ratio $V_n^{\star}(\lambda)/V_n(\lambda)$ to 1 when n is large enough.

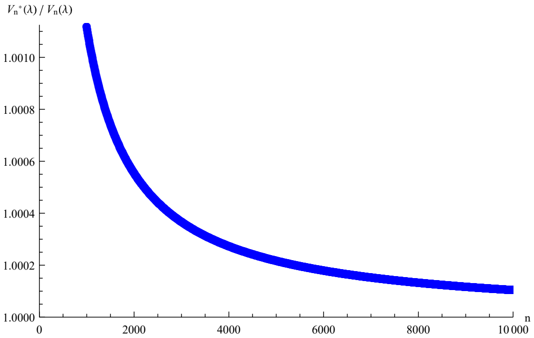

Figure 1. A plot illustrating the convergence tendency of the ratio $V_n^{\star}(\lambda)/V_n(\lambda)$ to 1 when n is large enough (by randomly selecting $\lambda = \frac{1}{2}$).

Table shows the values of the combinatorial-type numbers $V_n(\lambda)$, their Stirling approximation $V_n^{\star}(\lambda)$, and their ratio $V_n^{\star}(\lambda)/V_n(\lambda)$ in the special case when $\lambda = \frac{1}{2}$.

Table . In the special case when $\lambda = \frac{1}{2}$, the combinatorial-type numbers $V_n(\lambda)$, their Stirling approximation $V_n^{\star}(\lambda)$, and their ratio $V_n^{\star}(\lambda)/V_n(\lambda)$.

n	$V_n(\lambda)$	$V_n^{\star}(\lambda)$	$V_n^{\star}(\lambda)/V_n(\lambda)$
1	8	18.0541	2.25675833419103
10	$-6.87964 \cdot 10^{7}$	$-7.66275 \cdot 10^{7}$	1.11383051275245
100	$-4.54590 \cdot 10^{87}$	$-4.59710 \cdot 10^{87}$	1.01126328412454
1000	$-8.76968 \cdot 10^{898}$	$-8.77955 \cdot 10^{898}$	1.00112513281542
10000	$-1.79187 \cdot 10^{9025}$	$-1.79207 \cdot 10^{9025}$	1.00011250132813
100000	$-7.11507 \cdot 10^{90301}$	$-7.11515 \cdot 10^{90301}$	1.00001125001328
1000000	$-2.19016 \cdot 10^{903081}$	$-2.19017 \cdot 10^{903081}$	1.00000112500013
10000000	$-5.28938 \cdot 10^{9030889}$	$-5.28938 \cdot 10^{9030889}$	1.00000011250000

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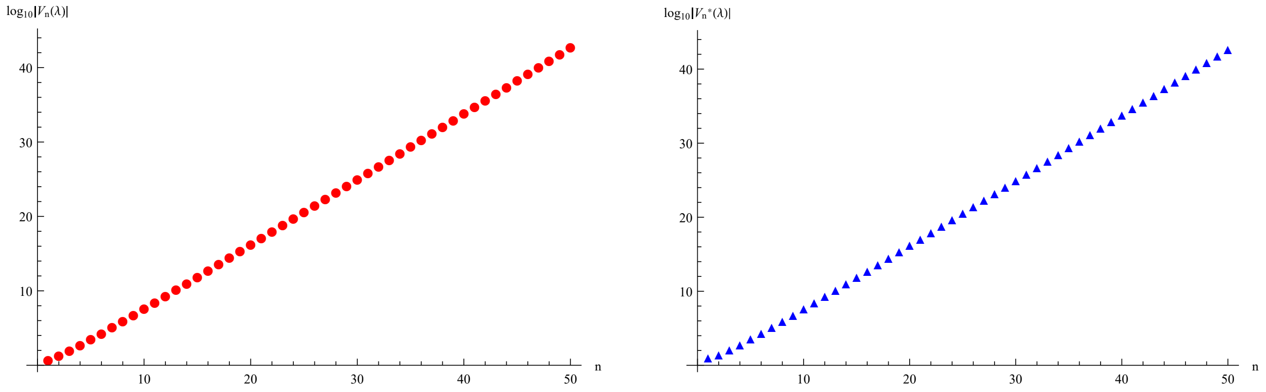
*Wolfram Research Inc., Mathematica Online (Wolfram Cloud). Champaign, IL, USA, 2020.

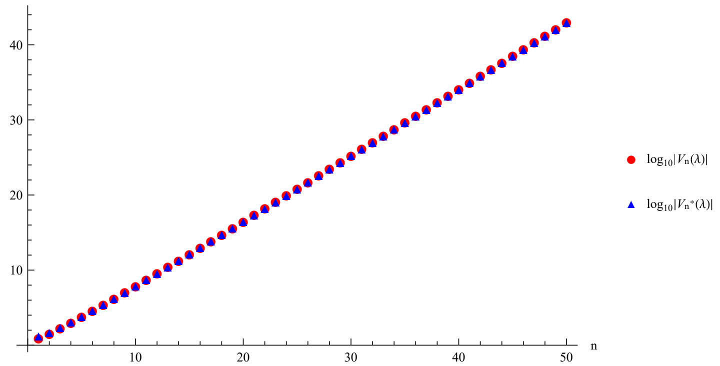
Remark 5.3 *Similar investigation for the Catalan numbers was conducted by Flajolet and Sedgewick in* [13].

By increasing the number of digits on the value of n, in addition to the values given in Table, Figure 2 provides the comparison of the absolute value of the numbers $V_n(\lambda)$ (represented by red filled circles) with the absolute value of its Stirling approximation $V_n^{\star}(\lambda)$ (represented by blue filled triangles) by a plot of their logarithms with base 10 versus n.



(a) Plot of $\log_{10}|V_n(\lambda)|$ versus n.

(b) Plot of $\log_{10}|V_n^{\star}(\lambda)|$ versus n.



(c) Combination of (a) and (b).

Figure 2. The comparison of the absolute value of the numbers $V_n(\lambda)$ (represented by red filled circles) with the absolute value of its Stirling approximation $V_n^{\star}(\lambda)$ (represented by blue filled triangles) by a plot of their logarithms with base 10 versus n while $\lambda = \frac{1}{2}$.

Combining (2.6) and (5.3) with the following well-known relation:

$$\frac{C_n}{C_{n-1}} = \frac{4n-2}{n+1}; \qquad (n \geq 1) \tag{5.4}$$

(*cf.* [5], [28, pp. 109-110]), we get the following theorem:

Theorem 5.4 *Let* $n \in \mathbb{N}_0$ *and* $\lambda \neq 1$. *Then we have*

$$\frac{V_{n+1}(\lambda)}{V_n(\lambda)} = -\frac{8n+4}{n+2}\left(\frac{\lambda}{\lambda-1}\right)^2. \tag{5.5}$$

By using (5.5), we have

$$\lim_{n\to\infty}\frac{V_{n+1}(\lambda)}{V_n(\lambda)}=-8\left(\frac{\lambda}{\lambda-1}\right)^2 \tag{5.6}$$

which yields the following corollary:

Corollary 5.5 *Let n be sufficiently large and $\lambda\neq 1$. Then we have*

$$V_{n+1}(\lambda)\approx -8\left(\frac{\lambda}{\lambda-1}\right)^2 V_n(\lambda). \tag{5.7}$$

6. Further remarks and observations on the combinatorial-type numbers

In this section, we give some remarks and observations on the numbers $V_n(\lambda)$ together with the relationships of these numbers to other well-known special numbers and polynomials.

6.1. Observations on the relations arising from the combinatorial-type numbers

Here, we give some observations on the formulas arising from the relations among the numbers $V_n(\lambda)$ and other well-known special numbers such as the Catalan numbers, the Stirling numbers, the Apostol–Bernoulli numbers, the Apostol–Euler numbers, the Apostol–Genocchi numbers, the numbers $Y_n(\lambda)$ and the combinatorial numbers $y_6(n,k;\lambda,v)$. The results of this subsection have potentially a wide range of applications due to their connections with the Catalan numbers, which emerges in the solution of a number of combinatorial enumeration problems and some real-world problems.

Recall that the well-known Catalan numbers are expressed by many different ways and among others some of which are given as follows:

$$C_n=\binom{2n}{n}-\binom{2n}{n+1};\quad (n\in\mathbb{N}_0) \tag{6.1}$$

$$C_n=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}^2;\quad (n\in\mathbb{N}_0) \tag{6.2}$$

and

$$C_n=\frac{1}{(n+1)!}\prod_{j=1}^{n}(4j-2);\quad (n\in\mathbb{N}) \tag{6.3}$$

(*cf.* [5], [9], [28], [35]).

In the next, by the combination of (2.6) respectively with (6.1), (6.2) and (6.3), observe that some computational formulas for the numbers $V_n(\lambda)$ are derived as in the following corollaries:

Corollary 6.1 *Let $n\in\mathbb{N}_0$ and $\lambda\neq 1$. Then we have*

$$V_n(\lambda)=(-1)^n\left[\binom{2n}{n}-\binom{2n}{n+1}\right]\frac{2^{n+1}\lambda^{2n}}{(\lambda-1)^{2n+1}}. \tag{6.4}$$

Corollary 6.2 *Let* $n \in \mathbb{N}_0$ *and* $\lambda \neq 1$*. Then we have*

$$V_n(\lambda) = \frac{(-1)^n 2^{n+1}}{(n+1)} \frac{\lambda^{2n}}{(\lambda-1)^{2n+1}} \sum_{k=0}^{n} \binom{n}{k}^2. \tag{6.5}$$

Corollary 6.3 *Let* $n \in \mathbb{N}$ *and* $\lambda \neq 1$*. Then we have*

$$V_n(\lambda) = \frac{(-1)^n 2^{n+1}}{(n+1)!} \frac{\lambda^{2n}}{(\lambda-1)^{2n+1}} \prod_{j=1}^{n} (4j-2). \tag{6.6}$$

Remark 6.4 *Combining* (6.5) *with* (1.19) *yields*

$$V_n(\lambda) = \frac{(-1)^n 2^{n+1} \lambda^{2n}}{(n+1)(\lambda-1)^{2n+1}} y_6(0,n;1,2), \tag{6.7}$$

which is an important indicator that the numbers $V_n(\lambda)$ *are associated with some families of special numbers and polynomials. Because, it is known from the work of Simsek* [43] *that the numbers* $y_6(n,k;\lambda,v)$ *are in relation to some families of special numbers and polynomials such as the Franel numbers, the Bernoulli numbers, the Euler numbers, the Stirling numbers, the Changhee numbers, the Daehee numbers, the Legendre polynomials, the Michael Vowe polynomials, the Mirimanoff polynomials, Golombek type polynomials, and others (see, for details,* [43]*). In that case, in connection with the aboves, the numbers* $V_n(\lambda)$ *are directly related to the relevant numbers and polynomials. These relationships show how strong the numbers* $V_n(\lambda)$ *are and have the potential to be used in many different areas.*

In addition to the above observation, combining (2.6) and (1.18) with (1.14) yields a relation between the numbers $V_n(\lambda)$ and the numbers $Y_n(\lambda)$ given by the following corollary:

Corollary 6.5 *Let* $n \in \mathbb{N}_0$ *and* $\lambda \neq 1$*. Then we have*

$$Y_n(\lambda) = \frac{(n+1)!(n!)^2(\lambda-1)^n}{(2n)!2^n} V_n(\lambda). \tag{6.8}$$

Remark 6.6 *With the combination of* (1.18) *with* (6.8)*, we also have the following identity containing the numbers* $V_n(\lambda)$*, the numbers* $Y_n(\lambda)$ *and the Catalan numbers:*

$$Y_n(\lambda) = n! \left(\frac{\lambda-1}{2}\right)^n \frac{V_n(\lambda)}{C_n}. \tag{6.9}$$

Theorem 6.7 *Let* $m \in \mathbb{N}$ *and* $\lambda \neq 0, 1$*. Then we have*

$$\mathcal{B}_m(\lambda) = \frac{m}{2} \sum_{n=0}^{m-1} \left(\frac{\lambda-1}{2\lambda}\right)^n \frac{(n+1)!(n!)^2}{(2n)!} S_2(m-1,n) V_n(\lambda). \tag{6.10}$$

Proof Combining (1.17) with (6.8) yields the assertion of Theorem 6.7. □

Remark 6.8 *Combining* (6.10) *with* (1.18), *Theorem* 6.7 *can also be written as follows:*

$$\mathcal{B}_m(\lambda) = \frac{m}{2}\sum_{n=0}^{m-1} n!\left(\frac{\lambda-1}{2\lambda}\right)^n \frac{S_2(m-1,n)\,V_n(\lambda)}{C_n}. \tag{6.11}$$

By combining (6.11) respectively with the equations (1.10) and (1.12), we have the following corollaries:

Corollary 6.9 *Let* $m \in \mathbb{N}_0$ *and* $\lambda \neq 0, -1$. *Then we have*

$$\mathcal{E}_m(\lambda) = -\sum_{n=0}^{m} n!\left(\frac{\lambda+1}{2\lambda}\right)^n \frac{S_2(m,n)\,V_n(-\lambda)}{C_n}, \tag{6.12}$$

Corollary 6.10 *Let* $m \in \mathbb{N}$ *and* $\lambda \neq 0, -1$. *Then we have*

$$\mathcal{G}_m(\lambda) = -m\sum_{n=0}^{m-1} n!\left(\frac{\lambda+1}{2\lambda}\right)^n \frac{S_2(m-1,n)\,V_n(-\lambda)}{C_n}. \tag{6.13}$$

Theorem 6.11 *Let* $m \in \mathbb{N}_0$ *and* $\lambda \neq -1$. *Then we have*

$$V_m(-\lambda) = -\frac{C_m}{m!}\left(\frac{2\lambda}{\lambda+1}\right)^m \sum_{n=0}^{m} \mathcal{E}_n(\lambda)\,S_1(m,n). \tag{6.14}$$

Proof Combining (1.15) with (6.8), we

$$V_m(-\lambda) = -\frac{(2m)!}{(m+1)!\,(m!)^2}\left(\frac{2\lambda}{\lambda+1}\right)^m \sum_{n=0}^{m} \mathcal{E}_n(\lambda)\,S_1(m,n). \tag{6.15}$$

By using (1.18) in the above equation, we arrive at the assertion of Theorem 6.11. □

Combining (6.14) respectively with the equations (1.10) and (1.11), we also get the following corollaries that give expressions of the numbers $V_m(\lambda)$ in terms of not only the Apostol-Bernoulli numbers, but also the Apostol-Genocchi numbers:

Corollary 6.12 *Let* $m \in \mathbb{N}_0$ *and* $\lambda \neq 1$. *Then we have*

$$V_m(\lambda) = \frac{2C_m}{m!}\left(\frac{2\lambda}{\lambda-1}\right)^m \sum_{n=0}^{m} \frac{\mathcal{B}_{n+1}(\lambda)\,S_1(m,n)}{n+1}. \tag{6.16}$$

Corollary 6.13 *Let* $m \in \mathbb{N}_0$ *and* $\lambda \neq 1$. *Then we have*

$$V_m(\lambda) = -\frac{C_m}{m!}\left(\frac{2\lambda}{\lambda-1}\right)^m \sum_{n=0}^{m} \frac{\mathcal{G}_{n+1}(\lambda)\,S_1(m,n)}{n+1}. \tag{6.17}$$

Remark 6.14 *Substituting* $\lambda = 1$ *into Theorem* 6.11, *we have*

$$V_m(-1) = -\frac{C_m}{m!}\sum_{n=0}^{m}\mathcal{E}_n(1)\,S_1(m,n). \tag{6.18}$$

Combining the above equation with

$$Y_m(-1) = (-1)^{m+1}\sum_{n=0}^{m}\mathcal{E}_n(1)\,S_1(m,n), \tag{6.19}$$

(*cf.* [52, Remark 1]), *we have*

$$V_m(-1) = (-1)^m\frac{C_m}{m!}Y_m(-1). \tag{6.20}$$

On the other hand, it is known that

$$Y_m(-1) = (-1)^{m+1}Ch_m, \tag{6.21}$$

where Ch_m *denotes the Changhee numbers* (*cf.* [52, Eq.(31)]). *Thus, using the above equality in* (6.20) *yields*

$$V_m(-1) = -\frac{C_m}{m!}Ch_m. \tag{6.22}$$

6.2. Observations on the relations arising from the Riemann zeta function and other well-known special numbers

Here, for the numbers $V_n(\lambda)$, we derive another computation formulas containing the Riemann zeta function, the Dirichlet series of the Möbius function and the Bernoulli numbers.

In order to derive the aforementioned computation formulas, we first need to recall the following definitions and relations:

The Riemann zeta function is defined by

$$\begin{aligned}\zeta(s) &= \sum_{m=1}^{\infty}\frac{1}{m^s} && (6.23)\\ &= \prod_{p\text{ prime}}\left(1-\frac{1}{p^s}\right)^{-1} && (6.24)\end{aligned}$$

where $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ (*cf.* [50]; and see also cited references therein).

Notice that the well-known relation between the Möbius function and the Riemann zeta function is given by the following Dirichlet series (*cf.* [2]):

$$\frac{1}{\zeta(s)} = \sum_{m=1}^{\infty}\frac{\mu(m)}{m^s} \tag{6.25}$$

as usual by the Euler's product formula given in (6.24) so that the Möbius function $\mu(m)$ is defined by

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1,\\ (-1)^v & \text{if } m \text{ is a square-free integer with } v \text{ distinct prime factors},\\ 0 & \text{if } m \text{ has a squared prime factor,}\end{cases}$$

(*cf.* [2]).

It is known from the work of Betts [4, Eq.(21)] that

$$C_n = \frac{\pi^{2n} r_n^{4n}}{n!(n+1)!\zeta(2n)}, \tag{6.26}$$

where

$$r_n = \sqrt{4n}2^{2n-1}|B_{2n}|$$

is the radius of a hypersphere in Euclidean dimension $4n$ and B_{2n} is the $2n^{th}$ Bernoulli number so that

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|,$$

(*cf.* [4]; also see [50]).

Combining (2.6) with (6.26), we get the following theorem which gives the numbers $V_n(\lambda)$ in terms of the Riemann zeta function.

Theorem 6.15 *Let $n \in \mathbb{N}$ and $\lambda \neq 1$. Then we have*

$$V_n(\lambda) = (-1)^n \frac{\pi^{2n} r_n^{4n} 2^{n+1}}{n!(n+1)!\zeta(2n)} \frac{\lambda^{2n}}{(\lambda-1)^{2n+1}}. \tag{6.27}$$

By substituting $s = 2n$ into (6.24), and then combining the final equation with (6.27), we get the following theorem:

Theorem 6.16 *Let $n \in \mathbb{N}$ and $\lambda \neq 1$. Then we have*

$$V_n(\lambda) = (-1)^n \frac{\pi^{2n} r_n^{4n} 2^{n+1}}{n!(n+1)!} \frac{\lambda^{2n}}{(\lambda-1)^{2n+1}} \prod_{p \ prime} \left(1 - \frac{1}{p^{2n}}\right). \tag{6.28}$$

Remark 6.17 *By using the fact that*

$$\zeta(4) = \frac{\pi^4}{90}$$

(cf.[†]), and the Euler's product formula, we investigate the equations (6.27) and (6.28) in the special case when $n = 2$ and $\lambda = \frac{1}{2}$ while primes are varying from $p = 2$ to $p = 71$ in (6.28). In that case, we have

$$\begin{aligned} V_2\left(\frac{1}{2}\right) &= (-1)^2 \frac{\pi^4(2^3\sqrt{8}|B_4|)^8 2^3}{2!3!\zeta(4)} \frac{\left(\frac{1}{2}\right)^4}{\left(-\frac{1}{2}\right)^5} \\ &= -12.5687200248743 \\ &\approx (-1)^2 \frac{\pi^4(2^3\sqrt{8}|B_4|)^8 2^3}{2!3!} \frac{\left(\frac{1}{2}\right)^4}{\left(-\frac{1}{2}\right)^5} \prod_{p=2}^{p=71} \left(1 - \frac{1}{p^4}\right) \\ &= -12.5687223075243. \end{aligned}$$

Thus, it can be seen from the above examination that even if only the first 20 prime numbers are considered, the difference between the calculated numbers is very small.

[†] Sloane NJA. The On-Line Encyclopedia of Integer Sequences. Sequence A013662.

By substituting $s = 2n$ into (6.25), and then combining the final equation with (6.27), we get the following theorem which includes a formula for the numbers $V_n(\lambda)$ including the Dirichlet series involving the Möbius function.

Theorem 6.18 *Let* $n \in \mathbb{N}$ *and* $\lambda \neq 1$. *Then we have*

$$V_n(\lambda) = (-1)^n \frac{\pi^{2n} r_n^{4n} 2^{n+1}}{n!(n+1)!} \frac{\lambda^{2n}}{(\lambda-1)^{2n+1}} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2n}}. \tag{6.29}$$

6.3. Observation on the difference between the numbers $V_n(\lambda)$ and the Catalan numbers

Here, in order to give an observation on the difference between the numbers $V_n(\lambda)$ and the Catalan numbers, we analyze the formula given in the equation (2.6) in detail with a graph given by Figure 3:

In order to illustrate where the Catalan numbers and the numbers $V_n(\lambda)$ differ from each other, we coded (2.6) by Wolfram programming language in Mathematica, and then we provide Figure 3 including a comparison of the absolute value of the numbers $V_n(\lambda)$ (represented by red filled circles) with the Catalan numbers C_n (represented by blue filled triangles) by plots of their logarithms with base 10 versus n while $\lambda = \frac{1}{2}$. Since all of the numbers $V_n(\lambda)$ are not positive and the Catalan numbers are all positive integers, for making comparison, the absolute value of the numbers $V_n(\lambda)$ is considered due to the domain of the logarithm function (for details, see the following figure).

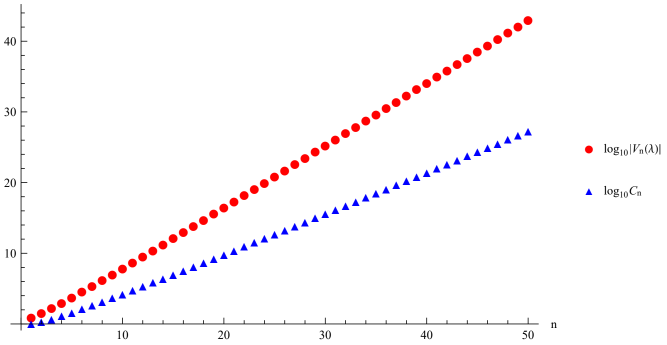


Figure 3. The comparison of the absolute value of the numbers $V_n(\lambda)$ (represented by red filled circles) with the Catalan numbers C_n (represented by blue filled triangles) by plots of their logarithms with base 10 versus n while $\lambda = \frac{1}{2}$.

6.4. Some inequalities for the numbers $V_n(\lambda)$

Here, we present some remarks and observations on some inequalities for the combinatorial-type numbers.

Let Ω be any simple closed contour surrounding the origin and $w \in \mathbb{C}$. Then, it is well-known that

$$2\pi i \binom{2n}{n} = \int_{\Omega} \frac{(1+w)^{2n}}{w^{n+1}} \mathrm{d}w \tag{6.30}$$

which, when Ω is a unit circle, reduces to

$$\binom{2n}{n} \leq 2^{2n} \tag{6.31}$$

(*cf.* [3]).

By combining (1.18) and (2.6) with (6.31), we get an inequality for the combinatorial-type numbers by the following theorem:

Theorem 6.19 *Let* $n\in\mathbb{N}_0$ *and* $\lambda\neq 1$. *Then we have*

$$V_n(\lambda)\leq\frac{(-1)^n 2^{3n+1}\lambda^{2n}}{(n+1)(\lambda-1)^{2n+1}}. \tag{6.32}$$

On the other hand, it is known that the Catalan numbers satisfy the following inequality:

$$C_n\geq\frac{2^{2n-1}}{(n+1)\sqrt{n}} \tag{6.33}$$

(*cf.* [40, Corollary 6.3, p. 1154]). If we combine (2.6) with (6.33), then we get another inequality for the combinatorial-type numbers by the following theorem:

Theorem 6.20 *Let* $n\in\mathbb{N}$ *and* $\lambda\neq 1$. *Then we have*

$$V_n(\lambda)\geq\frac{(-1)^n 2^{3n}\lambda^{2n}}{(n+1)\sqrt{n}(\lambda-1)^{2n+1}}. \tag{6.34}$$

By combining Theorem 6.19 with Theorem 6.20, we have the following corollary including the lower and upper bound for the numbers $V_n(\lambda)$:

Corollary 6.21 *Let* $n\in\mathbb{N}$ *and* $\lambda\neq 1$. *Then we have*

$$\frac{(-1)^n 2^{3n}\lambda^{2n}}{(n+1)\sqrt{n}(\lambda-1)^{2n+1}}\leq V_n(\lambda)\leq\frac{(-1)^n 2^{3n+1}\lambda^{2n}}{(n+1)(\lambda-1)^{2n+1}}. \tag{6.35}$$

7. Conclusion

In this paper, by using the methods of generating function, we have derived several formulas for a certain class of combinatorial-type numbers and polynomials. With the aid of these formulas, these numbers and polynomials have also been evaluated. By applying not only the p-adic integration methods but also the Riemann integral to the aforementioned combinatorial-type polynomials with multivariables, numerous formulas have been obtained that may have interest to researchers working on pure and applied mathematics. In addition, an investigation on an approximation for these numbers have been given by the aid of the Stirling's approximation for factorials. With the implementation of the results related to this approximation in Mathematica by Wolfram programming language, we have presented, Table and Figure 1–3 which give us numerical evaluations and illustrations on the approximation for the aforementioned combinatorial-type numbers and their Stirling's approximation. Some remarks and observations on the combinatorial-type numbers have been given together with the relationships of these numbers to other well-known special numbers and polynomials. These observations have yielded some computation formulas containing the Dirichlet series involving the Möbius function, the Bernoulli numbers, the Catalan numbers, the Stirling numbers, the Apostol–Bernoulli numbers, the Apostol–Euler numbers, the Apostol–Genocchi numbers and some kinds of combinatorial numbers. Finally, for the combinatorial-type numbers, some inequalities have been provided. To put it briefly and to interpret, the majority of the results obtained in this article are of a nature that can shed light on many areas of mathematics and computer science, especially computational science and engineering.

Acknowledgment

Second author of the present paper was supported by the Scientific Research Project Administration of Akdeniz University.

References

[1] Apostol TM. On the Lerch zeta function. Pacific Journal of Mathematics 1951; 1: 161-167. doi: 10.2140/pjm.1951.1.161

[2] Apostol TM. Introduction to Analytic Number Theory. New York-Heidelberg-Berlin: Springer-Verlag, 1976.

[3] Bak J, Newman DJ. Complex Analysis (Third Edition). New York, NY, USA: Springer, 2010.

[4] Betts RJ. A uniform convergent series for $\zeta(s)$ and closed formulas that including Catalan numbers. Preprint on ArXiv 2010; arXiv:1008.0387v3

[5] Bona M. Introduction to Enumerative Combinatorics. New York, NY, USA: The McGraw-Hill Companies Inc. 2007.

[6] Charalambides CA. Enumerative Combinatorics. London, New York: Chapman and Hall/Crc Press Company, 2002.

[7] Choi J. Note on Apostol-Daehee polynomials and numbers. Far East Journal of Mathematical Sciences 2017; 101 (8): 1845-1857. doi: 10.17654/MS101081845

[8] Comtet L. Advanced Combinatorics. Dordrecht-Holland/Boston-U.S.A.: D. Reidel Publication Company, 1974.

[9] Conway JH, Guy RK. In the Book of Numbers. New York, NY, USA: Springer-Verlag 1996.

[10] Cossali GE. A common generating function for Catalan numbers and other integer sequences. Journal of Integer Sequences 2003; 6: Article 03.1.8.

[11] Djordjević GB, Milovanović GV. Special Classes of Polynomials. Serbia: University of Nis, Faculty of Technology, 2014.

[12] Dolgy DV, Jang G-W, Kim DS, Kim T. Explicit expressions for Catalan-Daehee numbers. Proceedings of the Jangjeon Mathematical Society 2017; 20 (1): 1-9. doi: 10.23001/pjms2017.20.1.1

[13] Flajolet P, Sedgewick R. Analytic Combinatorics. Cambridge: Cambridge University Press 2009.

[14] Graham RL, Knuth DE, Patashnik O. Concrete Mathematics: A foundation for computer science (Second Edition). Addison-Wesley Publishing Company, 1994.

[15] Hilton P, Pedersen J. Catalan numbers, their generalization, and their uses. The Mathematical Intelligencer 1991; 13 (2): 64-75. doi: 10.1007/BF03024089

[16] Jordan C. Calculus of Finite Differences (2nd ed.), New York: Chelsea Publishing Company 1950.

[17] Kim DS, Kim T. Daehee numbers and polynomials. Applied Mathematical Sciences 2013; 7 (120): 5969-5976. doi: 10.12988/ams.2013.39535

[18] Kim DS, Kim T. A new approach to Catalan numbers using differential equations. Russian Journal of Mathematical Physics 2017; 24 (4): 465-475. doi: 10.1134/S1061920817040057

[19] Kim DS, Kim T, Seo J. A note on Changhee numbers and polynomials. Advanced Studies in Theoretical Physics 2013; 7: 993-1003. doi: 10.12988/astp.2013.39117

[20] Kim DS, Kim T, Kwon HI. Identities of some special mixed-type polynomials. Preprint on ArXiv 2014; arXiv:1406.2124v1

[21] Kim T. q-Volkenborn integration. Russian Journal of Mathematical Physics 2002; 19: 288-299.

[22] Kim T. q-Euler numbers and polynomials associated with p-adic q-integral and basic q-zeta function. Trends Mathematics (Information Center for Mathematical Sciences) 2006; 9: 7-12.

[23] Kim T. A note on Catalan numbers associated with p-adic integral on $\mathbb{Z}_p$. Proceedings of the Jangjeon Mathematical Society, 2016; 19 (3): 493-501.

[24] Kim T. An invariant p-adic q-integral on $\mathbb{Z}_p$. Applied Mathematics Letters 2008; 21: 105-108. doi: 10.1016/j.aml.2006.11.011

[25] Kim T, Kim DS. Differential equations associated with Catalan-Daehee numbers and their applications. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2017; 111: 1071-1081. doi: 10.1007/s13398-016-0349-4

[26] Kim T, Kim DS, Dolgy DV, Seo J-J. Bernoulli polynomials of the second kind and their identities arising from umbral calculus. Journal of Nonlinear Science and Applications 2016; 9: 860-869. doi: 10.22436/jnsa.009.03.14

[27] Kim T. Rim S-H, Dolgy DV, Pyo S-S. Explicit expression for symmetric identities of w-Catalan-Daehee polynomials. Notes on Number Theory and Discrete Mathematics 2018; 24 (4): 99-111. doi: 10.7546/nntdm.2018.24.4.99-111

[28] Koshy T. Catalan numbers with applications. Oxford, New York: Oxford University Press 2009.

[29] Kucukoglu I, Simsek B, Simsek Y. An approach to negative hypergeometric distribution by generating function for special numbers and polynomials. Turkish Journal of Mathematics 2019; 43: 2337-2353. doi: 10.3906/mat-1906-6

[30] Kucukoglu I, Simsek B, Simsek Y. New classes of Catalan-type numbers and polynomials with their applications related to p-adic integrals and computational algorithms. Turkish Journal of Mathematics 2020; 44: 2337-2355. doi: 10.3906/mat-2008-24

[31] Luo QM, Srivastava HM. Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind. Applied Mathematics and Computation 2011; 217: 5702-5728. doi: 10.1016/j.amc.2010.12.048

[32] Merlini D, Sprugnoli R, Verri MC. The Cauchy numbers. Discrete Mathematics 2006; 306 (16): 1906-1920. doi: 10.1016/j.disc.2006.03.065

[33] Qi F, Guo BN. Sums of infinite power series whose coefficients involve products of the Catalan-Qi numbers. Montes Taurus Journal of Pure and Applied Mathematics 2019; 1 (2): 1-12. Article ID: MTJPAM-D-19-00007

[34] Roman S. The Umbral Calculus. New York: Dover Publ. Inc. 2005.

[35] Roman S. An Introduction to Catalan Numbers. Birkhauser 2015.

[36] Schikhof WH. Ultrametric Calculus: An Introduction to p-adic Analysis. Cambridge: Cambridge Stud. Adv. Math. 4, Cambridge University Press 1984.

[37] Simsek Y. Generating functions for generalized Stirling type numbers, array type polynomials, Eulerian type polynomials and their applications. Fixed Point Theory and Applications 2013; 87: 343-355. doi: 10.1186/1687-1812-2013-87

[38] Simsek Y. Analysis of the Bernstein basis functions: an approach to combinatorial sums involving binomial coefficients and Catalan numbers. Mathematical Methods in the Applied Sciences 2015; 38 (14): 3007-3021. doi: 10.1002/mma.3276

[39] Simsek Y. Identities on the Changhee numbers and Apostol-type Daehee polynomials. Advanced Studies in Contemporary Mathematics (Kyungshang) 2017; 27 (2): 199-212.

[40] Simsek Y. Combinatorial sums and binomial identities associated with the Beta-type polynomials. Hacettepe Journal of Mathematics and Statistics 2018; 47 (5): 1144-1155. doi: 10.15672/HJMS.2017.505

[41] Simsek Y. Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q-integrals. Turkish Journal of Mathematics 2018; 42: 557-577. doi: 10.3906/mat-1703-114

[42] Simsek Y. Peters type polynomials and numbers and their generating functions: Approach with p-adic integral method. Mathematical Methods in the Applied Sciences 2019; 42 (18): 7030-7046. doi: 10.1002/mma.5807

[43] Simsek Y. Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers. Journal of Mathematical Analysis and Applications 2019; 477: 1328-1352. doi: 10.1016/j.jmaa.2019.05.015

[44] Simsek Y. Explicit formulas for p-adic integrals: Approach to p-adic distributions and some families of special numbers and polynomials. Montes Taurus Journal of Pure and Applied Mathematics 2019; 1 (1): 1-76. Article ID: MTJPAM-D-19-00005

[45] Simsek Y. Interpolation functions for new classes special numbers and polynomials via applications of p-adic integrals and derivative operator. Montes Taurus Journal of Pure and Applied Mathematics 2021; 3 (1): 38-61. Article ID: MTJPAM-D-20-00000

[46] Simsek Y, Yardimci A. Applications on the Apostol-Daehee numbers and polynomials associated with special numbers, polynomials, and p-adic integrals. Advances in Difference Equations 2016; Article number: 308. doi: 10.1186/s13662-016-1041-x

[47] Srivastava HM. Some formulas for the Bernoulli and Euler polynomials at rational arguments. Mathematical Proceedings of the Cambridge Philosophical Society 2000; 129: 77-84. doi: 10.1017/S0305004100004412

[48] Srivastava HM. Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials. Applied Mathematics & Information Sciences 2011; 5: 390-444.

[49] Srivastava HM, Choi J. Series Associated with the Zeta and Related Functions. Dordrecht, the Netherlands: Kluwer Academic Publishers, 2001.

[50] Srivastava HM, Choi J. Zeta and q-zeta functions and associated series and integrals. Amsterdam, London and New York: Elsevier Science Publishers, 2012.

[51] Srivastava HM, Kim T, Simsek Y. q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series. Russian Journal of Mathematical Physics 2005; 12: 241-268.

[52] Srivastava HM, Kucukoglu I, Simsek Y. Partial differential equations for a new family of numbers and polynomials unifying the Apostol-type numbers and the Apostol-type polynomials. Journal of Number Theory 2017; 181: 117-146. doi: 10.1016/j.jnt.2017.05.008