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Exact gain of Brillouin fiber amplifiers in the regime of pump depletion

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Abstract: Thanks to the LambertW function, the gain of a Brillouin fiber amplifier operating in the depleted pump regime is solved numerically using the Newton-Raphson method. Unlike traditional boundary value problem solvers, which rely on shooting or relaxation techniques and often pose challenges such as high computational complexity and convergence issues, the Newton-Raphson method offers a simpler and more efficient numerical approach. The result obtained can be used to optimize the performance of Brillouin fiber amplifiers.

1. Introduction

Since the first demonstration of light amplification in optical fibers with stimulated Brillouin scattering (SBS), the Brillouin fiber amplifier (BFA) configuration [1] has been implemented in a wide range of applications, such as microwave photonics [2], shape-adjustable narrowband optical filtering [3], radio- over-fiber technology [4], generation of millimeter-wave signals [5], exploiting as tunable slow-light delay buffers [6] and fiber optic sensing [7].

In the BFA configuration, interaction between the pump and the Stokes wave due to SBS is described by a system of nonlinear ordinary differential equations (ODEs) [8]. The system of ODEs for BFAs has well-defined boundary conditions: $P_p(0) = P_0$ and $P_s(L) = P_{seed}$, with L being the fiber length. Such a mathematical problem is known as the two-point boundary value problem (BVP). Since the nonlinear interaction can be maintained over a large distance. The gain of the Brillouin amplifier exponentially increases with pump power but tends to saturate eventually if pump power reaches some critical value. In this case, the conventionally used undepleted pump approximation (UPA) [8] significantly overestimates the Brillouin gain.

Several attempts have been undertaken to find a general analytical solution to SBS equations in a lossy medium [9–13]. A conserved quantity of the system of ODEs reduces the problem to a single equation [9]. However, it can only be integrated numerically. In another approach [10], it was proposed to reverse the sign of the loss term in one of the ODEs to make the approximate set of equations integrable. This results in a system of two transcendental equations to be solved numerically with no closed-form solution possible. In Ref [11], a closed-form approximate analytical formula for the gain of a BFA operating in the depleted pump regime is derived. Here, the material loss of the medium, which is a strong effect in optical fibers is considered. In our previous works [12,13], we derived a highly accurate approximate formula for the gain of a BFA operating in the depleted pump regime. The analytical solution is obtained for three distinct regions based on the pump power level: the weak pump regime, the high-gain regime, and the saturation regime. Notably, for the saturation regime, the LambertW function was also employed to accurately describe the nonlinear behavior of the amplifier.

In this work, the gain of a BFA operating in the depleted pump regime is solved numerically using the Newton-Raphson method rather than BVP solvers. Here, the system of nonlinear ODEs is first transformed into an implicit integral form, which involves the LambertW function. The use of the LambertW function allows the implicit integral to be easily differentiated with

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respect to the unknowns, leading to a root-finding problem that can be solved using the Newton-Raphson method, as demonstrated in Ref. [14]. The LambertW function, which originates from the pioneering work of J. H. Lambert and was later refined by Euler [15,16], is now implemented in various software packages with high precision [17]. This special function plays a key role in transforming the BVP problem into a root finding problem, thereby enabling the effective application of the Newton-Raphson method. Unlike traditional BVP solvers, which rely on shooting or relaxation techniques and often pose challenges such as high computational complexity and convergence issues, the Newton-Raphson method offers a simpler and more efficient numerical approach.

The proposed method also works well for a Raman fiber amplifier, which is described by a similar set of ODEs [14]. However, unlike Raman scattering, where the interaction is primarily with optical phonons and the gain spectrum is relatively broad (~13 THz), Brillouin scattering involves interaction with acoustic phonons, resulting in a much narrower gain bandwidth (~25 MHz) and a smaller frequency shift (~10 GHz). This makes Brillouin systems more susceptible to challenges such as acoustic mode dominance and pump depletion at lower power levels [18,19].

2. Model description

The coupled ODEs for the evolution of the pump P_p and Stokes P_s power can be written as [8,11]

$$dP_p/dz = -\gamma P_p P_s - \alpha P_p, \qquad dP_S/dz = -\gamma P_p P_s + \alpha P_s \tag{1}$$

where $\gamma = g_B/A_{ao}$ is the peak SBS efficiency for fibers with a single dominant acoustic mode, $0 \leq z \leq 0$ is the propagation distance along the optical fiber of the total length L, α is the fiber loss coefficient, g_B is the Brillouin gain coefficient, and A_{ao} is the acousto-optic effective area that for some optical fibers, it differs from the optical effective area A_{eff}. Here we assume a seed wave launched from the rear end of the fiber. Here, we assume the Stokes wave is not initiated by noise, but it is launched from the rear end of the fiber. Then the known values of the input pump power $P_p(0)$ and the input Stokes power $P_s(L)$ are the boundary values.

For our analysis, it is convenient to introduce dimensionless quantities $\xi = z/L$, $w = P_p(z)/P_p(0)$, $u = P_s(z)/P_p(0)$, $a = \alpha L$, $k = \gamma P_p(0)L$, and $\varepsilon = P_s(\mathrm{L})/P_p(0)$ and rewrite Eq. (1) as

$$w_\xi = -kuw - aw, \qquad u_\xi = -kuw + au, \tag{2}$$

where the subscript denotes differentiation with respect to the normalized length $\xi(0 \leq \xi \leq 1)$, while k, a, and ε are the three system parameters that describe strength of the nonlinear interaction, loss, and power of the seed wave relative to the pump power, respectively. The boundary conditions now take the form $w(0) = 1$, $u(1) = \epsilon$

In our approach, we obtain conserved quantity C by solving Eqs. (2), $\int_{w(0)}^{w(\xi)} (-kw + a)\frac{dw}{w} = \int_{u(0)}^{u(\xi)} (-ku - a)\frac{du}{u}$ as $C(w, u) = \frac{a}{k} ln\left(\frac{1}{w(\xi)u(\xi)}\right) + (w(\xi) - u(\xi))$, and represent the unknown u in terms of w and C i.e., (Appendix A)

$$u(\xi, C) = \frac{a}{k}\boldsymbol{W}\left(\frac{k}{aw(\xi)}exp\left(-\frac{k}{a}(C - w(\xi))\right)\right) \tag{3}$$

where $\boldsymbol{W}$ is the LambertW function which satisfies $\boldsymbol{W}(x) \cdot e^{\boldsymbol{W}(x)} = x$. Substitution of Eq. (3) into first of Eq. (2) results in:

$$\int_{w(\xi)}^{w(0)} \frac{dw}{aw\left[\boldsymbol{W}\left[\frac{k}{aw}exp\left(-\frac{k}{a}(C - w)\right)\right] + 1\right]} = \xi \tag{4}$$

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Equation (4) results in a transcendental equation to be solved numerically with no closed-form solution possible where conserved quantity $C(w(1)) = \frac{a}{k} ln\left(\frac{1}{w(1)u(1)}\right) + (w(1) - u(1))$. Thus, integral function is defined,

$$F[w(1)] = \int_{w(1)}^{1} \frac{dw}{aw\left[\mathbf{W}\left[\frac{k}{aw} exp\left(-\frac{k}{a}\{C(w(1)) - w\}\right)\right] + 1\right]} - 1 \tag{5}$$

$$\begin{aligned} \frac{dF[w(1)]}{dw(1)} = \int_{w(1)}^{1} & \frac{\left(k - \frac{a}{w(1)}\right)\mathbf{W}\left[\frac{k}{aw} exp\left(-\frac{k}{a}\{C(w(1)) - w\}\right)\right] dw}{a^2 w\left\{\mathbf{W}\left[\frac{k}{aw} exp\left(-\frac{k}{a}\{C(w(1)) - w\}\right)\right] + 1\right\}^3} \\ & - \left\{aw(1)\left[\mathbf{W}\left[\frac{k}{aw(1)} exp\left(-\frac{k}{a}\{C[w(1)] - w(1)\}\right)\right] + 1\right]^{-1}\right\} \end{aligned} \tag{6}$$

Since the derivative is known, the Newton–Raphson method can be efficiently applied to find $w(1)$ by solving the equation $F[w(1)] = 0$. Then $u(0)$ can be obtained via $C = \frac{a}{k} ln\left(\frac{1}{u(0)}\right) + (1 - u(0))$. Here, initial value of $w(1) = 10^{-7}$ works well for all Pump and Stokes boundary values. Exact solutions are plotted in Fig. 1 together with the UPA solution where $w_{UPA}(\xi) = e^{-a\xi}$ and $u_{UPA}(\xi) = \varepsilon \cdot exp\left[a(\xi - 1) + \frac{k}{a}(e^{-a\xi} - e^{-a})\right]$.

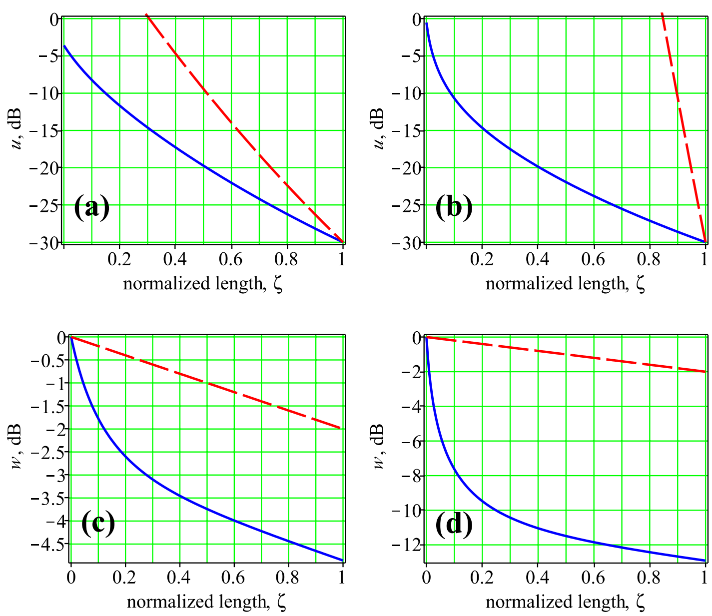

Fig. 1. Stokes wave P_s (a), (b) and Pump wave P_p (c),(d) as a function normalized fiber distance, ζ. In [(a), (c)], $P_{p0} = 10\ mW$, $P_{sL} = 10\ \mu W$, $(k = 14)$, in [(b), (d)] and $P_{p0} = 50\ mW$, $P_{sL} = 50\ \mu W$, $(k = 70)$. Thick solid curves, Numerical solution, dashed curves, UPA solution. In both figures, $L = 10\ km$, $\gamma = 0.14\ m^{-1}W^{-1}$, $\alpha = 0.20$ dB$/km$.

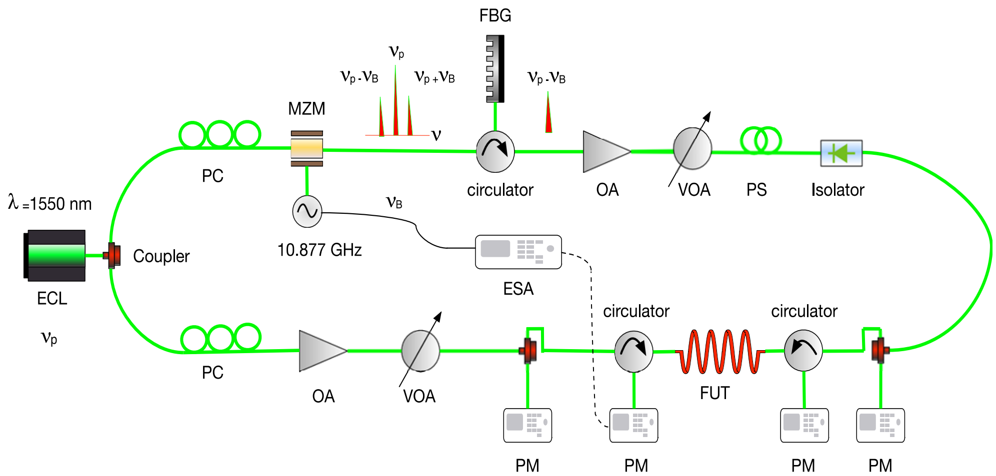

Fig. 2. Experimental setup for BFA measurements: ECL, external cavity laser, PC, polarization controller; MZM, Mach–Zehnder modulator; FBG, fiber Bragg grating; OA, optical amplifier; VOA, variable optical attenuator; PS: polarization scrambler; PM, powermeter; ESA, electrical spectrum analyzer and FUT, fiber under test.

3. Experimental setup

In the experimental setup (Fig. 2), external cavity laser (ECL) (SFL1550S-Thorlabs) with output power 40 mW, linewidth smaller than 50 kHz and $\lambda_p = 1550$ nm is split into pump and probe channels. In the pump side, laser was used together with an EDFA (DA-EYDFA -Dawnergy) to generate up to 200 mW of pump power. In the probe side, the stokes wave was obtained using a Mach–Zehnder modulator (MZM)(LN65S-FC-Thorlabs). The electrical spectrum analyzer (ESA) (FSV3000- Rohde & Schwarz) was used to determine the Brillouin frequency shift (ν_B) of the fiber and, correspondingly, the modulation frequency for the MZM (10.877 GHz for standard single-mode fiber). A fiber Bragg grating (FBG) in the reflecting regime was used to select a sideband of the MZM output. A Stokes-shifted probe is selected by FBG (FBG Reflector [1550 nm] At Grating Technologies). A polarization scrambler (POL-100K-Thorlabs) and isolator (IO-5-1550-VLP-Thorlabs) are also utilized before launching the probe signal into the fiber under test (FUT), which is a standard single-mode fiber (Corning SMF-28e Fiber, ITU G.652 compliant). The polarization scrambler averages out the polarization dependence on the Brillouin gain, while the isolator avoids the interference of pump in MZM. The polarization controller (PC)(FPC650-Thorlabs) placed at the output of the laser was used to maximize the detected power. Powermeters (PM400-Thorlabs) were used to monitor the input pump power P_{p0}, the transmitted pump power P_{pL}, the launched seed power P_{sL}, and the amplified Stokes power P_{s0}. Fiber with length of 10 km. were experimentally studied.

The experimental results are plotted in (Fig. 3) together with numerical predictions. From (Fig. 3(a), the nonlinear coefficient γ was estimated from the numerical fit of the measured transmitted pump power $P_p(L)$, while measured value of the fiber loss coefficient was $\alpha = 0.2$ dB/km. During measurement, polarization scrambler and laser source with the linewidth of 50 kHz is employed. The found value of $\gamma = 0.14\,\mathrm{m}^{-1}\,\mathrm{W}^{-1}$ which is very close the reported to the value in Ref. [11] was then used in computing the other curves.

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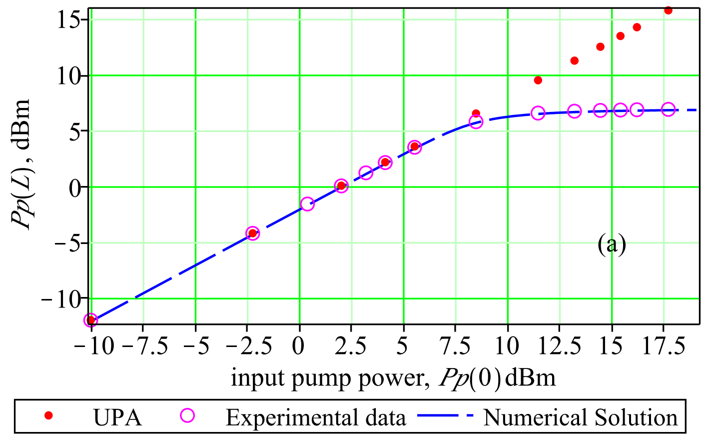

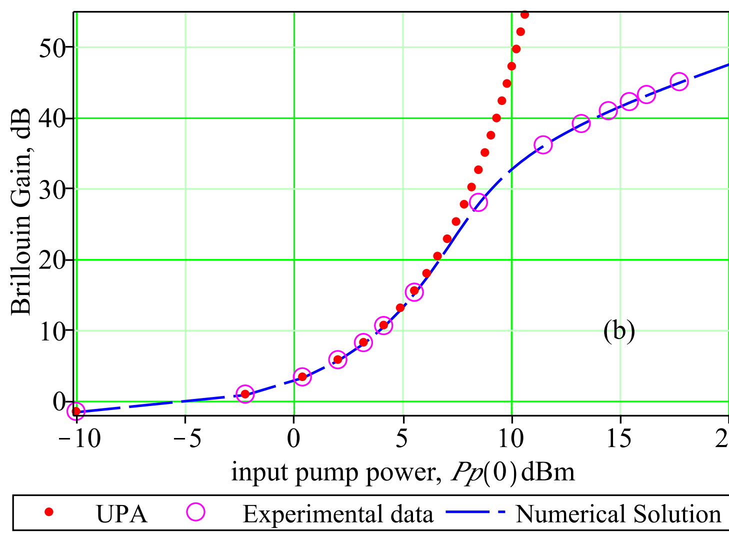

Fig. 3. (a) Transmitted pump power P_p(L) and (b) BFA gain versus the input pump power $P_p(0)$. Open circles, experimental data; dashed curves, numerical results. In both figures, $\gamma = 0.14\ m^{-1}W^{-1}$, $L = 10\ km$, $P_{SL} = -28\ dBm$, $\alpha = 0.20$ dB/km,

4. Conclusions

In conclusion, thanks to the LambertW function, the gain of a BFA operating in the depleted pump regime is solved numerically using the Newton-Raphson method rather than BVP solvers. This special function plays a key role in transforming the BVP problem into a root finding problem, thereby enabling the effective application of the Newton-Raphson method. Unlike traditional BVP solvers, which use shooting or relaxation techniques and often face challenges like high computational complexity and convergence issues, the Newton-Raphson method provides a much simpler and more efficient numerical approach. The result obtained can be used to optimize performance of Brillouin fiber amplifiers. Such amplifiers are widely used in microwave photonics, radio-over-fiber technology, and sensing. This method can also be straightforwardly extended to characterize Raman and doped-fiber amplifiers and lasers.

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Appendix A: expressions for Eq. (3) ($u(\xi, C)$)

$$C = \frac{a}{k}\ln\left(\frac{1}{wu}\right) + (w - u) \tag{7}$$

$$C = -\frac{a}{k}\ln(wu) + (w - u) \tag{8}$$

$$u - w + C = -\frac{a}{k}\ln(wu) \tag{9}$$

$$exp(u - w + C) = exp\left(-\frac{a}{k}\ln(wu)\right) \tag{10}$$

$$e^{(u-w+C)} = (wu)^{-\frac{a}{k}} \tag{11}$$

$$(wu)^{\frac{a}{k}} = e^{-(u-w+C)} \tag{12}$$

Lambert W function is defined as:

$$z = \boldsymbol{W}(z)e^{\boldsymbol{W}(z)} \tag{13}$$

To use LambertW function, we must transform Eq. (A6) into a new form:

$$(wu)^{\frac{a}{k}} = e^{-u+w-C} \tag{14}$$

$$wu = e^{-\frac{k}{a}u+\frac{k}{a}(w-C)} \tag{15}$$

$$wu = e^{-\frac{k}{a}u} \cdot e^{\frac{k}{a}(w-C)} \tag{16}$$

$$u \cdot e^{\frac{k}{a}u} = \frac{1}{w} \cdot e^{\frac{k}{a}(w-C)} \tag{17}$$

Let: $z = \frac{k}{a}u$

$$z \cdot e^{z} = \frac{k}{a} \cdot \frac{1}{w} \cdot e^{\frac{k}{a}(w-C)} \tag{18}$$

Now we can apply LambertW function.

$$z = \boldsymbol{W}\left(\frac{k}{a} \cdot \frac{1}{w} \cdot e^{\frac{k}{a}(w-C)}\right) \tag{19}$$

Since $z = \frac{k}{a}u$, we can obtain u as,

$$u = \frac{a}{k} \cdot \boldsymbol{W}\left(\frac{k}{a \cdot w} \cdot e^{-\frac{k}{a}(C-w)}\right) \tag{20}$$

Disclosures. The authors declare no conflicts of interest.

Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

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References

1. N. A. Olsson and J. P. van der Ziel, "Cancellation of fiber loss by semiconductor laser pumped Brillouin amplification at 1.5 µm," Appl. Phys. Lett. **48**(20), 1329–1330 (1986).
2. D. Loayssa, M. J. Benito, and Garde, "Applications of optical carrier Brillouin processing to microwave photonics," Opt. Fiber Technol. **8**(1), 24–42 (2002).
3. T. Tanemura, Y. Takushima, and K. Kikuchi, "Narrowband optical filter, with a variable transmission spectrum, using stimulated Brillouin scattering in optical fiber," Opt. Lett. **27**(17), 1552–1554 (2002).
4. M. J. LaGasse, W. Charczenko, M. C. Hamilton, *et al.*, "Optical carrier filtering for high dynamic range fibre optic links," Electron. Lett. **30**(25), 2157–2158 (1994).
5. T. Schneider, M. Junker, and D. Hannover, "Generation of millimetre- wave signals by stimulated Brillouin scattering for radio over fibre systems," Electron. Lett. **40**(23), 1500–1502 (2004).
6. L. Xing, L. Zhan, S. Luo, *et al.*, "High-power low-noise fiber Brillouin amplifier for tunable slow-light delay buffer," IEEE J. Quantum Electron. **44**(12), 1133–1138 (2008).
7. D. Culverhouse, F. Frahi, C.N. Pannell, *et al.*, "Potential of stimulated Brillouin scattering as sensing mechanism for distributed temperature sensor," Electron. Lett. **25**(14), 913–915 (1989).
8. R. W. Boyd, *Nonlinear Optics*, 3rd ed., (Academic 2007), Chap. 9.
9. L. Chen and X. Bao, "Analytical and numerical solutions for steady state stimulated Brillouin scattering in a single-mode fiber," Opt. Commun. **152**(1-3), 65–70 (1998).
10. Le Sébastien, Pierre Floch, and Cambon, "Theoretical evaluation of the Brillouin threshold and the steady-state Brillouin equations in standard single-mode optical fibers," J. Opt. Soc. Am. A **20**(6), 1132–1137 (2003).
11. S. Kobyakov, M. Darmanyan, Sauer, *et al.*, "High-Gain Brillouin amplification: an analytical approach," Opt. Lett. **31**(13), 1960–1962 (2006).
12. Fikri Serdar Gökhan, Hasan Göktaş, Volker J. Sorger, *et al.*, "Analytical approach of Brillouin amplification over threshold," Appl. Opt. **57**(4), 607–611 (2018).
13. Fikri Serdar Gökhan and Hasan Göktaş, "Analytical approach to calculate the gain of Brillouin fiber amplifiers in the regime of pump depletion," Appl. Opt. **58**(27), 7628–7635 (2019).
14. M. Santagiustina, "Exact integral solution of saturation and depletion in forward and backward optical fiber Raman amplifiers," Opt. Lett. **32**(20), 3023–3025 (2007).
15. J.H. Lambert, "Observationes variae in mathesin puram," Acta Helvetica, physicomathematico-anatomico-botanico-medica, 3, Basel, (1758) 128–168.
16. L. Euler, "De formulis exponentialibus replicatis," Leonhardi Euleri Opera Omnia, Ser. 1, Opera Mathematica **15**, 268–297 (1927).
17. https://www.maplesoft.com/support/help/Maple/view.aspx?path=LambertW.
18. A. Kobyakov, S. Kumar, D. Q. Chowdhury, *et al.*, "Design concept for optical fibers with enhanced SBS threshold," Opt. Express **13**(14), 5338–5346 (2005).
19. L. Thévenaz, "Slow and fast light in optical fibres," Nat. Photonics **2**(8), 474–481 (2008).